MAR513-Lec.8: Non-Hydrostatic Dynamics

What is the difference between hydrostatic and non-hydrostatic?

Why are the current ocean models based on hydrostatic approximation?

(1) Numerical consideration

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_o}\frac{\partial P}{\partial x} + \frac{\partial}{\partial z}(K_m \frac{\partial u}{\partial z}) + F_u \tag{8.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_o}\frac{\partial P}{\partial y} + \frac{\partial}{\partial z}(K_m \frac{\partial v}{\partial z}) + F_v \tag{8.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_o}\frac{\partial P}{\partial z} - \frac{\rho}{\rho_o}g + \frac{\partial}{\partial z}(K_m \frac{\partial w}{\partial z}) + F_w \tag{8.4}$$

There is no explicit time-marching equation for pressure P !

The solution form of pressure P is obtained by substituting Eqs. (8.2)-(8.4) into Eq. (8.1) and it generally can be written as:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = RHS$$

Too expensive to compute!

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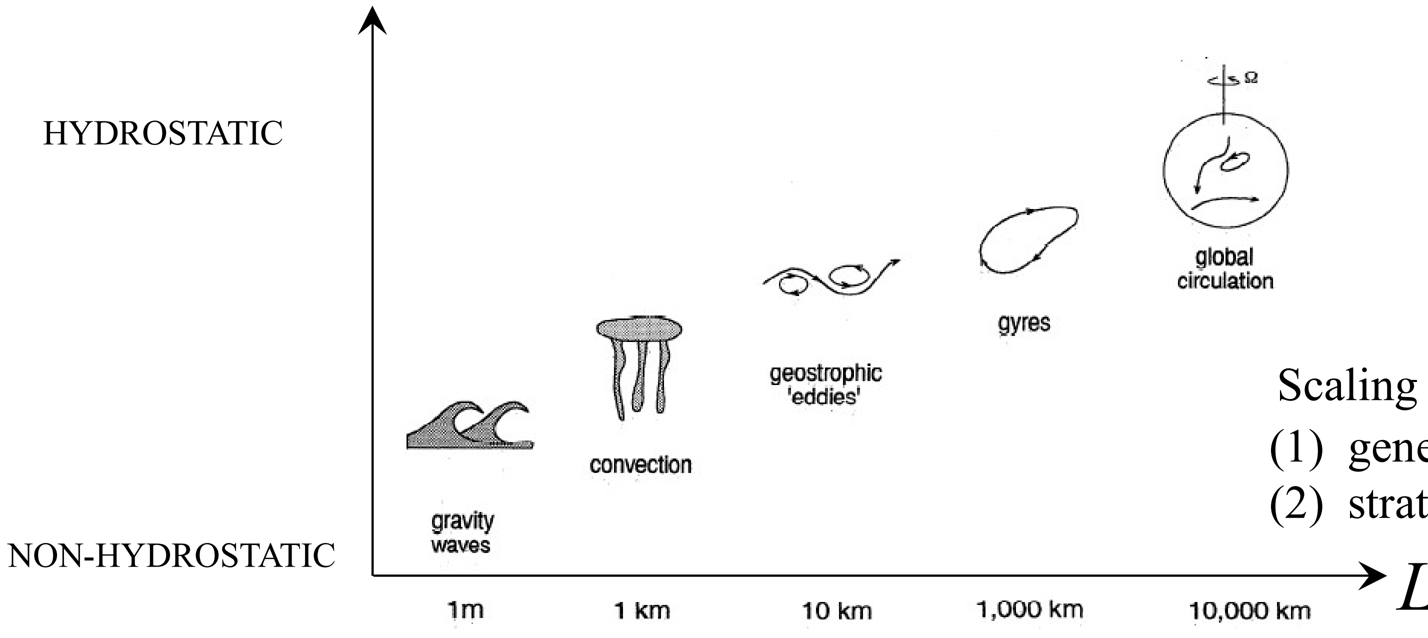
(2) Physical consideration

The large-scale circulations computed by ocean models are basically hydrostatic !

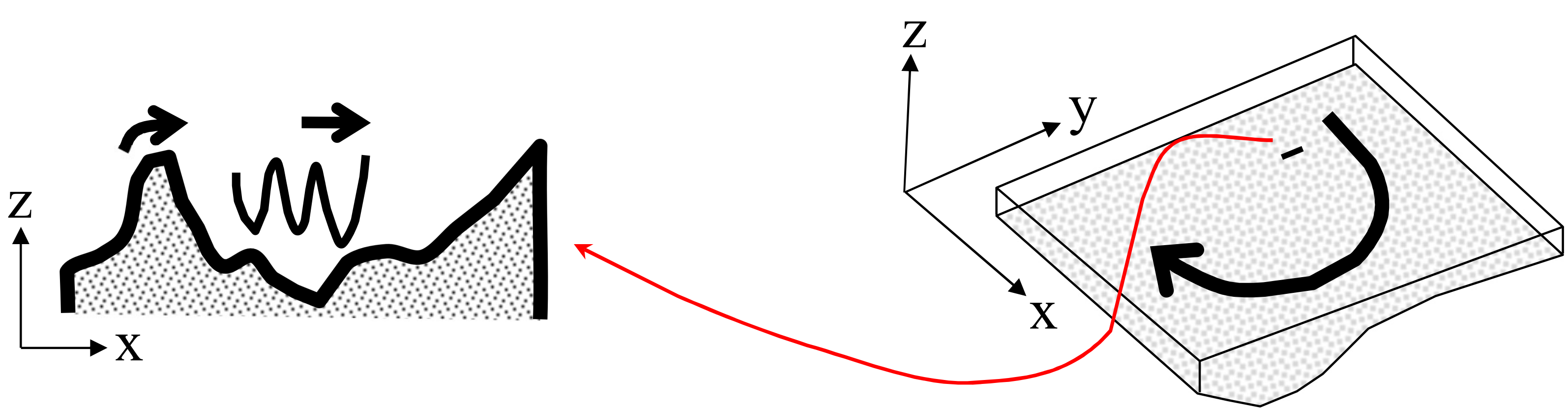


Scaling analysis:

(1) general, $H/L \leq 1$;

(2) stratification, $(U/L)/N \leq 1$

"Adopted from Marshall et al. (1997)"



MAR513-Lec.8: Non-Hydrostatic Dynamics

What the hydrostatic approximation means for Eqs. (8.1)-(8.4)?

Non-hydrostatic

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_o}\frac{\partial P}{\partial x}+\frac{\partial}{\partial z}(K_m\frac{\partial u}{\partial z})+F_u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_o}\frac{\partial P}{\partial y}+\frac{\partial}{\partial z}(K_m\frac{\partial v}{\partial z})+F_v$$

$$\cancel{\frac{\partial w}{\partial t}}+\cancel{u\frac{\partial w}{\partial x}}+\cancel{v\frac{\partial w}{\partial y}}+\cancel{w\frac{\partial w}{\partial z}}=-\frac{1}{\rho_o}\frac{\partial P}{\partial z}-\frac{\rho}{\rho_o}g+\cancel{\frac{\partial}{\partial z}(K_m\frac{\partial w}{\partial z})}+\cancel{F_w}$$

Hydrostatic

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_o}\frac{\partial P}{\partial x}+\frac{\partial}{\partial z}(K_m\frac{\partial u}{\partial z})+F_u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_o}\frac{\partial P}{\partial y}+\frac{\partial}{\partial z}(K_m\frac{\partial v}{\partial z})+F_v$$

$$\frac{\partial P}{\partial z}=-\rho g$$

If defined: $P=P_a+P_H+\cancel{q}$

P_a: atmospheric pressure

P_H: hydrostatic pressure and

$$\frac{\partial p_H}{\partial z}=-\rho g\Rightarrow p_H=\rho_0 g\zeta+g\int_z^0\rho dz$$

q : non-hydrostatic pressure

$$\frac{\partial\zeta}{\partial t}+\frac{\partial uD}{\partial x}+\frac{\partial vD}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_o}\frac{\partial\zeta}{\partial x}-\frac{1}{\rho_o}\frac{\partial B}{\partial x}+\frac{\partial}{\partial z}(K_m\frac{\partial u}{\partial z})+F_u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_o}\frac{\partial\zeta}{\partial y}-\frac{1}{\rho_o}\frac{\partial B}{\partial y}+\frac{\partial}{\partial z}(K_m\frac{\partial v}{\partial z})+F_v$$

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How to solve non-hydrostatic equations (8.1)-(8.4) ?

(1) Streamfunction/vorticity method
see the work of Shen and Evans (2004, JCP) and Scotti et al. (2007, JGR).
But this method only works for 2D case!

(2) Artificial compressibility method (Chorin, 1967, JCP)

$$\partial_t u_i = -\partial_i p + F_i$$

$$(\partial_t \rho) + \partial_j u_j = 0$$

$$p = \rho / \delta$$

δ : the artificial compressibility

(3) Fractional-step method (Chorin, 1968, Math. Comp.)
A nice discussions of the methods including the projection, pressure correction and iteration method is given by Armfield and Street (2002, Int. J. Numer. Methods Fluids).

MAR513-Lec.8: Non-Hydrostatic Dynamics

The derivation of fractional-step method:

Step1: time split of the half-discretized momentum equations

$$\frac{u^{n+1}-u^n}{\Delta t} = -flux_u - \frac{1}{\rho}\frac{\partial q}{\partial x}$$
$$\frac{v^{n+1}-v^n}{\Delta t} = -flux_v - \frac{1}{\rho}\frac{\partial q}{\partial y}$$
$$\frac{w^{n+1}-w^n}{\Delta t} = -flux_w - \frac{1}{\rho}\frac{\partial q}{\partial z}$$

Projection method

$$\frac{u^*-u^n}{\Delta t} = -flux_u^n$$
$$\frac{v^*-v^n}{\Delta t} = -flux_v^n$$
$$\frac{w^*-w^n}{\Delta t} = -flux_w^n$$

$$\frac{u^{n+1}-u^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q^{n+1}}{\partial x}$$
$$\frac{v^{n+1}-v^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q^{n+1}}{\partial y}$$
$$\frac{w^{n+1}-w^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q^{n+1}}{\partial z}$$

Pressure correction (iterative) method

$$\frac{u^*-u^n}{\Delta t} = -flux_u^n - \frac{1}{\rho}\frac{\partial q^n}{\partial x}$$
$$\frac{v^*-v^n}{\Delta t} = -flux_v^n - \frac{1}{\rho}\frac{\partial q^n}{\partial y}$$
$$\frac{w^*-w^n}{\Delta t} = -flux_w^n - \frac{1}{\rho}\frac{\partial q^n}{\partial z}$$

$$\frac{u^{n+1}-u^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q'}{\partial x}$$
$$\frac{v^{n+1}-v^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q'}{\partial y}$$
$$\frac{w^{n+1}-w^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial q'}{\partial z}$$

$$q^{n+1} = q^n + q'$$

MAR513-Lec.8: Non-Hydrostatic Dynamics

Step2: predict the intermediate velocity field

Projection method

$$u^* = u^n - \Delta t \cdot flux_u^n$$

$$v^* = v^n - \Delta t \cdot flux_v^n$$

$$w^* = w^n - \Delta t \cdot flux_w^n$$

Pressure correction (iterative) method

$$u^* = u^n - \Delta t \cdot flux_u^n - \frac{\Delta t}{\rho}\frac{\partial q^n}{\partial x}$$

$$v^* = v^n - \Delta t \cdot flux_v^n - \frac{\Delta t}{\rho}\frac{\partial q^n}{\partial y}$$

$$w^* = w^n - \Delta t \cdot flux_w^n - \frac{\Delta t}{\rho}\frac{\partial q^n}{\partial z}$$

A key issue here is the B.C. for the intermediate velocities!

It was demonstrated that using physical conditions for intermediate velocities at B.C. will cause the projection method be first-order accuracy in time, while the pressure correction (iterative) method is second-order time accuracy.

MAR513-Lec.8: Non-Hydrostatic Dynamics

Step3: Solve the non-hydrostatic pressure and correct the velocity field

Projection method

$$u^{n+1} = u^* - \frac{\Delta t}{\rho}\frac{\partial q^{n+1}}{\partial x}$$

$$v^{n+1} = v^* - \frac{\Delta t}{\rho}\frac{\partial q^{n+1}}{\partial y}$$

$$w^{n+1} = w^* - \frac{\Delta t}{\rho}\frac{\partial q^{n+1}}{\partial z}$$

Pressure correction (iterative) method

$$u^{n+1} = u^* - \frac{\Delta t}{\rho}\frac{\partial q'}{\partial x}$$

$$v^{n+1} = v^* - \frac{\Delta t}{\rho}\frac{\partial q'}{\partial y}$$

$$w^{n+1} = w^* - \frac{\Delta t}{\rho}\frac{\partial q'}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q^{n+1} = \frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q' = \frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right)$$

$$q^{n+1} = q^n + q'$$

MAR513-Lec.8: Non-Hydrostatic Dynamics

How to develop a non-hydrostatic ocean model?

(1) There are many well-developed and validated hydrostatic ocean models such as FVCOM, MITgcm, POM, ROMs which are free available. Can we build a non-hydrostatic ocean model based on these models or we have to start from the very beginning?

(2) Choose structured or unstructured grid? What is the matrix properties of the discretized non-hydrostatic pressure Poisson equation? How to solve it efficiently?

(3) The fractional-step is originated from and applied extensively in CFD. But in ocean modeling, we are facing additional issues such as surface moving boundary, vertical sigma coordinate and split-mode time-stepping method. How to adjust this method for these issues.

(4) How to design a non-hydrostatic algorithm which is mass conserved?

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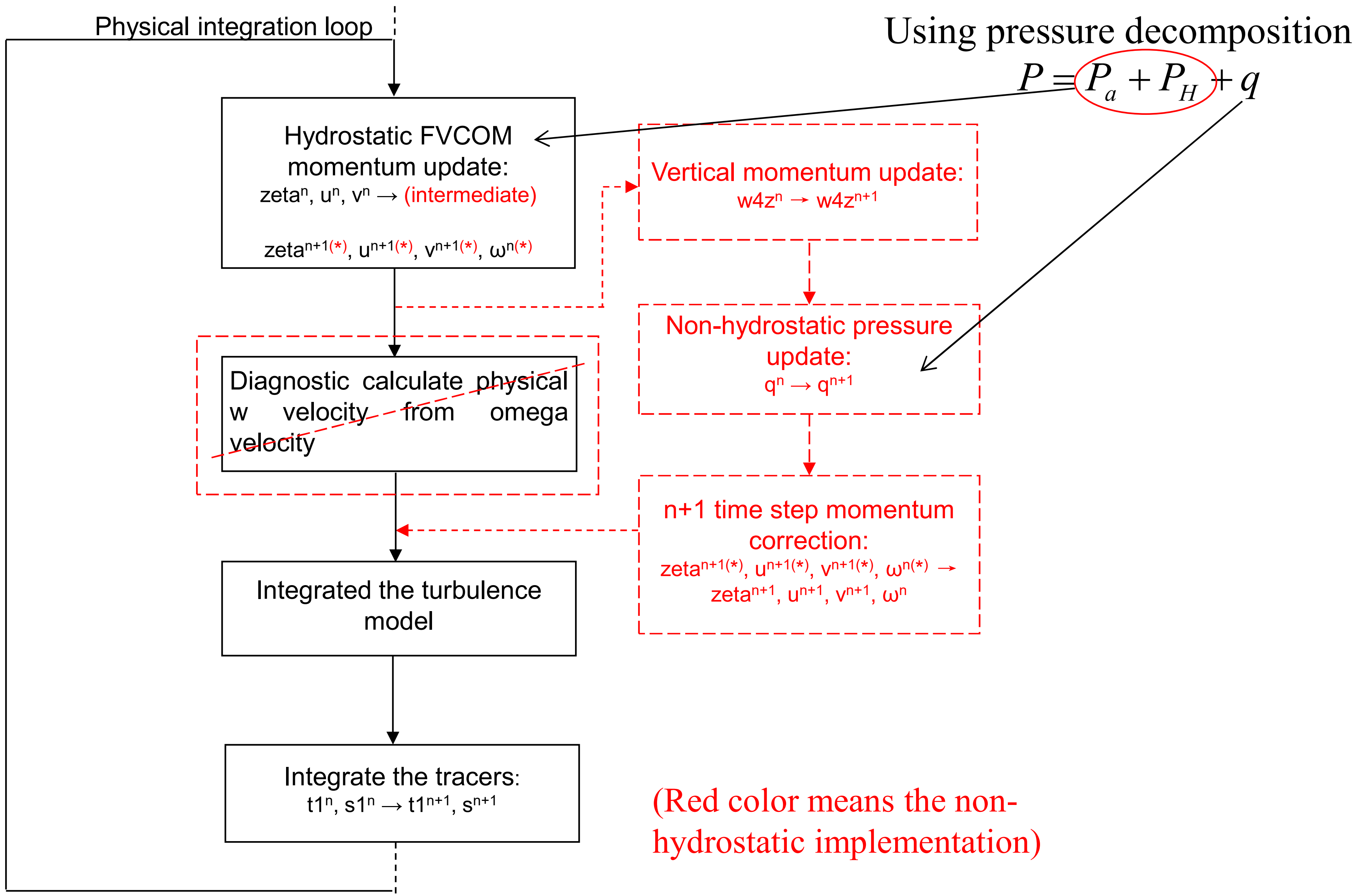
We use non-hydrostatic FVCOM (FVCOM-NH) as an example:



MAR513-Lec.8: Non-Hydrostatic Dynamics

(1) The non-hydrostatic primitive equations in the sigma coordinate:

$$\frac{\partial uD}{\partial t}+\frac{\partial u^2 D}{\partial x}+\frac{\partial uvD}{\partial y}+\frac{\partial u\omega}{\partial \sigma}-fvD=-gD\frac{\partial \zeta}{\partial x}-\frac{D}{\rho_o}\frac{\partial p_a}{\partial x}-\frac{gD}{\rho_o}[\int_\sigma^0 D\frac{\partial \rho}{\partial x}d\sigma-\frac{\partial D}{\partial x}\int_\sigma^0 \sigma\frac{\partial \rho}{\partial \sigma}d\sigma]$$
$$-\frac{D}{\rho_0}[\frac{\partial q}{\partial x}+\frac{\partial \sigma}{\partial x}\frac{\partial q}{\partial \sigma}]+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial u}{\partial \sigma})+DF_u \quad (8.5)$$

$$\frac{\partial vD}{\partial t}+\frac{\partial uvD}{\partial x}+\frac{\partial v^2 D}{\partial y}+\frac{\partial v\omega}{\partial \sigma}+fuD=-gD\frac{\partial \zeta}{\partial y}-\frac{D}{\rho_o}\frac{\partial p_a}{\partial y}-\frac{gD}{\rho_o}[\int_\sigma^0 D\frac{\partial \rho}{\partial y}d\sigma-\frac{\partial D}{\partial y}\int_\sigma^0 \sigma\frac{\partial \rho}{\partial \sigma}d\sigma]$$
$$-\frac{D}{\rho_o}[\frac{\partial q}{\partial y}+\frac{\partial \sigma}{\partial y}\frac{\partial q}{\partial \sigma}]+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial v}{\partial \sigma})+DF_v \quad (8.6)$$

$$\frac{\partial wD}{\partial t}+\frac{\partial uwD}{\partial x}+\frac{\partial vwD}{\partial y}+\frac{\partial w\omega}{\partial \sigma}=-\frac{1}{\rho_o}\frac{\partial q}{\partial \sigma}+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial w}{\partial \sigma})+DF_w \quad (8.7)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \sigma}{\partial x}\frac{\partial u}{\partial \sigma}+\frac{\partial \sigma}{\partial y}\frac{\partial v}{\partial \sigma}+\frac{1}{D}\frac{\partial w}{\partial \sigma}=0$$ Continuity equation for divergent free (8.8)

$$\frac{\partial \zeta}{\partial t}+\frac{\partial uD}{\partial x}+\frac{\partial vD}{\partial y}+\frac{\partial \omega}{\partial \sigma}=0$$ Continuity equation for free surface and omega velocity (8.9)

MAR513-Lec.8: Non-Hydrostatic Dynamics

(2) Fractional-step formula

$$
\begin{cases}
\dfrac{u^* D^* - u^n D^n}{\Delta t} = -F_x^n - a\dfrac{D}{\rho_o}\left(\dfrac{\partial q^n}{\partial x} + \dfrac{\partial \sigma}{\partial x}\dfrac{\partial q^n}{\partial \sigma}\right) + \dfrac{1}{D}\dfrac{\partial}{\partial \sigma}\left(K_m \dfrac{\partial u^*}{\partial \sigma}\right) & (8.10) \\[2ex]
\dfrac{v^* D^* - v^n D^n}{\Delta t} = -F_y^n - a\dfrac{D}{\rho_o}\left(\dfrac{\partial q^n}{\partial y} + \dfrac{\partial \sigma}{\partial y}\dfrac{\partial q^n}{\partial \sigma}\right) + \dfrac{1}{D}\dfrac{\partial}{\partial \sigma}\left(K_m \dfrac{\partial v^*}{\partial \sigma}\right) & (8.11) \\[2ex]
\dfrac{w^* D^* - w^n D^n}{\Delta t} = -F_z^n - a\dfrac{1}{\rho_o}\dfrac{\partial q^n}{\partial \sigma} + \dfrac{1}{D}\dfrac{\partial}{\partial \sigma}\left(K_m \dfrac{\partial w^*}{\partial \sigma}\right) & (8.12)
\end{cases}
$$

$$
\begin{cases}
\dfrac{u^{n+1} - u^*}{\Delta t} = -\dfrac{1}{\rho_o}\left(\dfrac{\partial q'}{\partial x} + \dfrac{\partial \sigma}{\partial x}\dfrac{\partial q'}{\partial \sigma}\right) & (8.13) \\[2ex]
\dfrac{v^{n+1} - v^*}{\Delta t} = -\dfrac{1}{\rho_o}\left(\dfrac{\partial q'}{\partial y} + \dfrac{\partial \sigma}{\partial y}\dfrac{\partial q'}{\partial \sigma}\right) & (8.14) \\[2ex]
\dfrac{w^{n+1} - w^*}{\Delta t} = -\dfrac{1}{\rho_o D}\dfrac{\partial q'}{\partial \sigma} & (8.15)
\end{cases}
$$

$$q^{n+1} = a \cdot q^n + q'$$

$a = 0$: projection

$a = 1$: pressure correction

MAR513-Lec.8: Non-Hydrostatic Dynamics

(3) Solve the intermediate free surface and velocities

- Split-mode explicit time stepping method

Step1: vertically integrated Eqs (8.5), (8.6) and (8.9) for $\zeta^*, \overline{u^*}, \overline{v^*}$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\overline{u}D)}{\partial x} + \frac{\partial(\overline{v}D)}{\partial y} = 0$$

$$\frac{\partial \overline{u}D}{\partial t} + \frac{\partial \overline{u}^2 D}{\partial x} + \frac{\partial \overline{u}\overline{v}D}{\partial y} - f\overline{v}D - D\overline{F}_u - G_x - \frac{\tau_{sx} - \tau_{bx}}{\rho_o}$$

$$= -gD\frac{\partial \zeta}{\partial x} - \frac{gD}{\rho_o}\{\int_{-1}^{0} \frac{\partial}{\partial x}(D\int_{\sigma}^{0} \rho \, d\sigma')d\sigma + \frac{\partial D}{\partial x}\int_{-1}^{0} \sigma \rho d\sigma\} + \underline{\int_{-1}^{0} [\frac{D}{\rho_o}(\frac{\partial q}{\partial x} + \frac{\partial \sigma}{\partial x}\frac{\partial q}{\partial \sigma})]d\sigma}$$

$$\frac{\partial \overline{v}D}{\partial t} + \frac{\partial \overline{u}\,\overline{v}D}{\partial x} + \frac{\partial \overline{v}^2 D}{\partial y} + f\overline{u}D - D\overline{F}_v - G_y - \frac{\tau_{sy} - \tau_{by}}{\rho_o}$$

$$= -gD\frac{\partial \zeta}{\partial y} - \frac{gD}{\rho_o}\{\int_{-1}^{0} \frac{\partial}{\partial y}(D\int_{\sigma}^{0} \rho d\sigma')d\sigma + \frac{\partial D}{\partial y}\int_{-1}^{0} \sigma \rho d\sigma\} + \underline{\int_{-1}^{0} [\frac{D}{\rho_o}(\frac{\partial q}{\partial y} + \frac{\partial \sigma}{\partial y}\frac{\partial q}{\partial \sigma})]d\sigma}$$

Step2: bring ζ^* into Eqs. (8.5)-(8.7) to solve u^*, v^*, w^*

MAR513-Lec.8: Non-Hydrostatic Dynamics

- Semi-implicit time stepping method

Step1: rewrite Eqs (8.5) - (8.6) into the semi-implicit form

$$\frac{\partial uD}{\partial t}+\frac{\partial u^2D}{\partial x}+\frac{\partial uvD}{\partial y}+\frac{\partial u\omega}{\partial \sigma}-fvD=-gD[(1-\theta)\frac{\partial \zeta^n}{\partial x}+\theta\frac{\partial \zeta^{n+1}}{\partial x}]-\frac{D}{\rho_o}\frac{\partial p_a}{\partial x}$$
$$-\frac{gD}{\rho_o}[\int_\sigma^0 D\frac{\partial \rho}{\partial x}d\sigma-\frac{\partial D}{\partial x}\int_\sigma^0 \sigma\frac{\partial \rho}{\partial \sigma}d\sigma]-\frac{D}{\rho_o}(\frac{\partial q}{\partial x}+\frac{\partial \sigma}{\partial x}\frac{\partial q}{\partial \sigma})+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial u}{\partial \sigma})+DF_u \quad (8.16)$$

$$\frac{\partial vD}{\partial t}+\frac{\partial uvD}{\partial x}+\frac{\partial v^2D}{\partial y}+\frac{\partial v\omega}{\partial \sigma}+fuD=-gD[(1-\theta)\frac{\partial \zeta^n}{\partial y}+\theta\frac{\partial \zeta^{n+1}}{\partial y}]-\frac{D}{\rho_o}\frac{\partial p_a}{\partial y}$$
$$-\frac{gD}{\rho_o}[\int_\sigma^0 D\frac{\partial \rho}{\partial y}d\sigma-\frac{\partial D}{\partial y}\int_\sigma^0 \sigma\frac{\partial \rho}{\partial \sigma}d\sigma]-\frac{D}{\rho_o}(\frac{\partial q}{\partial y}+\frac{\partial \sigma}{\partial y}\frac{\partial q}{\partial \sigma})+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial v}{\partial \sigma})+DF_v \quad (8.17)$$

or its simplified form

$$\frac{\partial uD}{\partial t}=XFLUX^n-g\theta D\frac{\partial \zeta^{n+1}}{\partial x}+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial u}{\partial \sigma}) \quad (8.18)$$

$$\frac{\partial vD}{\partial t}=YFLUX^n-g\theta D\frac{\partial \zeta^{n+1}}{\partial y}+\frac{1}{D}\frac{\partial}{\partial \sigma}(K_m\frac{\partial v}{\partial \sigma}) \quad (8.19)$$

MAR513-Lec.8: Non-Hydrostatic Dynamics

- Semi-implicit time stepping method

Step2: Integrating Eqs. (8.18)-(8.19) from $\sigma=-1$ to 0 yields

$$(\overline{u}D)^{n+1}=(\overline{u}D)^{n}+\Delta t\int_{-1}^{0}XFLUX^{n}d\sigma-g\theta D\Delta t\frac{\partial\zeta^{n+1}}{\partial x}+\Delta t\frac{\tau_{sx}^{n}-\tau_{bx}^{n}}{D} \quad (8.20)$$

$$(\overline{v}D)^{n+1}=(\overline{v}D)^{n}+\Delta t\int_{-1}^{0}YFLUX^{n}d\sigma-g\theta D\Delta t\frac{\partial\zeta^{n+1}}{\partial y}+\Delta t\frac{\tau_{sy}^{n}-\tau_{by}^{n}}{D} \quad (8.21)$$

and bring Eqs. (8.20)-(8.21) into vertically integrated semi-implicit continuity equation:

$$\frac{\partial\zeta}{\partial t}+(1-\theta)\frac{\partial(\overline{u}D)^{n}}{\partial x}+\theta\frac{\partial(\overline{u}D)^{n+1}}{\partial x}+(1-\theta)\frac{\partial(\overline{v}D)^{n}}{\partial y}+\theta\frac{\partial(\overline{v}D)^{n+1}}{\partial y}=0 \quad (8.22)$$

It will result in a 2D linear system for surface elevation

$$A_{2D}\zeta^{n+1}=B_{2D} \quad (8.23)$$

After solving intermediate free surface, again, we can bring ζ^{*} into Eqs. (8.18)-(8.19) and (8.7) to solve u^{*},v^{*},w^{*}

MAR513-Lec.8: Non-Hydrostatic Dynamics

(4) Solving non-hydrostatic pressure

Substitute Eqs. (8.13)-(8.15) into continuity equation (8.8) that will result in the non-hydrostatic pressure equation:

$$\frac{\partial^2 q'}{\partial x^2} + \frac{\partial^2 q'}{\partial y^2} + [(\frac{\partial \sigma}{\partial x})^2 + (\frac{\partial \sigma}{\partial y})^2 + \frac{1}{D^2}]\frac{\partial^2 q'}{\partial \sigma^2} + 2(\frac{\partial \sigma}{\partial x}\frac{\partial^2 q'}{\partial x \partial \sigma} + \frac{\partial \sigma}{\partial y}\frac{\partial^2 q'}{\partial y \partial \sigma}) + (\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2})\frac{\partial q'}{\partial \sigma}$$
$$= \frac{\rho_0}{\Delta t}(\frac{\partial u^*}{\partial x} + \frac{\partial \sigma}{\partial x}\frac{\partial u^*}{\partial \sigma} + \frac{\partial v^*}{\partial y} + \frac{\partial \sigma}{\partial y}\frac{\partial v^*}{\partial \sigma} + \frac{1}{D}\frac{\partial w^*}{\partial \sigma}) \qquad (8.24)$$

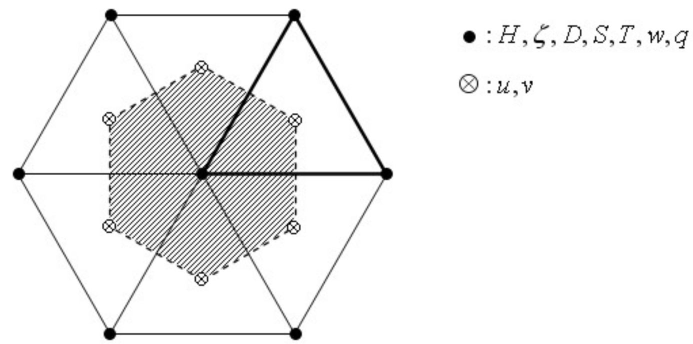
The boundary conditions for Eq. (8.24):

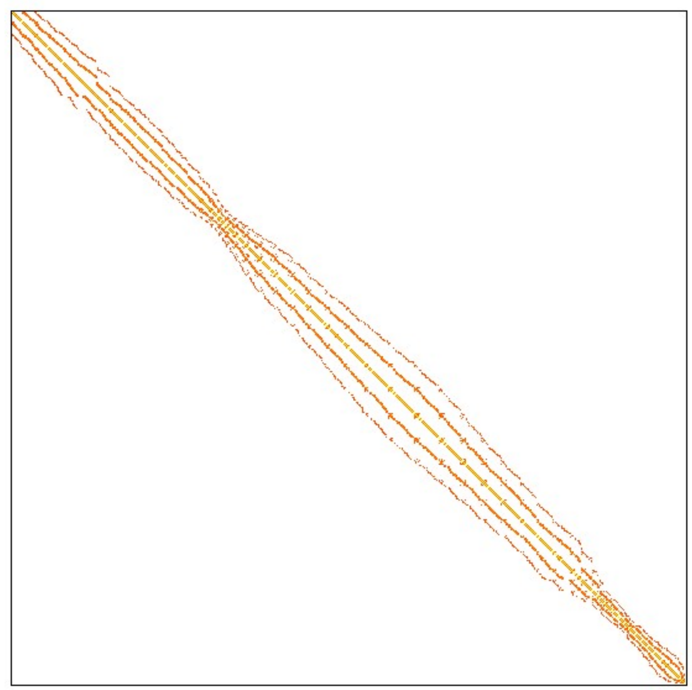
- at surface $\qquad q' = 0$
- at bottom $\qquad \frac{\partial q'}{\partial \sigma} = \frac{D \tan \beta}{(1+\tan^2 \beta)}\frac{\partial q'}{\partial n}$
- at lateral solid wall $\qquad \frac{\partial q'}{\partial n_h} = -(n_x \cdot \frac{\partial \sigma}{\partial x} + n_y \cdot \frac{\partial \sigma}{\partial y})\frac{\partial q'}{\partial \sigma}$
- at open boundary $\qquad$ let $u^* = u^{n+1}$, $v^* = v^{n+1}$, $w^* = w^{n+1}$ to derive a form of q'

MAR513-Lec.8: Non-Hydrostatic Dynamics

The discretization of Eq. (8.24) will results in a large sparse matrix: $Aq' = b$

To solve it, we employ a parallel sparse matrix solver library (PETSc) (*Balay et al*., 2007) and High Performance Preconditioners (HYPRE) software library (*Falgout and Yang*, 2002)





MAR513-Lec.8: Non-Hydrostatic Dynamics

(5) Correct the intermediate velocity field and free surface

Once obtain the n+1 time step q, it is easy to correct the intermediate velocity field based on Eqs (8.13)-(8.15):

$$u^{n+1} = u^{*} - \frac{\Delta t}{\rho_o}\left(\frac{\partial q'}{\partial x} + \frac{\partial \sigma}{\partial x}\frac{\partial q'}{\partial \sigma}\right) \qquad v^{n+1} = v^{*} - \frac{\Delta t}{\rho_o}\left(\frac{\partial q'}{\partial y} + \frac{\partial \sigma}{\partial y}\frac{\partial q'}{\partial \sigma}\right) \qquad w^{n+1} = w^{*} - \frac{\Delta t}{\rho_o D}\frac{\partial q'}{\partial \sigma}$$

For other non-hydrostatic ocean models, the n+1 time step integration is finished at this stage without further correcting the free surface (assuming the error is small!). But our numerical experiment indicate that this will cause free surface damping. So we also correct the intermediate free surface by

First compute $\bar{u}^{n+1} = \int_{-1}^{0} u^{n+1} d\sigma$ and $\bar{v}^{n+1} = \int_{-1}^{0} v^{n+1} d\sigma$

Then update ζ^{*} by: $\dfrac{\zeta^{n+1} - \zeta^{n}}{\Delta t} + \dfrac{\partial[\bar{u}^{n+1}(H + \zeta^{n+1})]}{\partial x} + \dfrac{\partial[\bar{v}^{n+1}(H + \zeta^{n+1})]}{\partial y} = 0$

MAR513-Lec.8: Non-Hydrostatic Dynamics

How we validate FVCOM-NH?

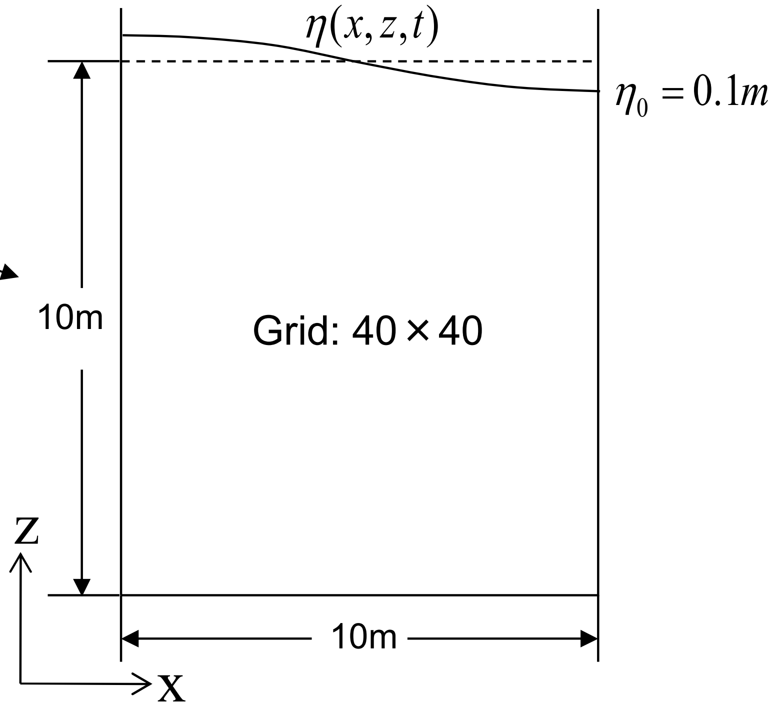
Test cases	Testing non-hydrostatic dynamics for	Validation method
Surface standing wave	Linear wave	Analytical solution
Surface solitary waves	Non-linear wave	Analytical solution (Grimshaw, 1971, Fenton, 1972), lab experiment (Madsen and Mei, 1969)
Lock-exchange flows	Internal flow	Analytical estimate (Turner, 1973) numerical simulation (Hartel et al, 2000)
Internal solitary wave breaking on slopes	Realistic flows	Lab experiment (Michallet and Ivey ,1999)

MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case1: surface standing wave

Setup a 2D problem by assuring no along y-direction gradient. The same approach was applied in later numerical tests

Vertical view



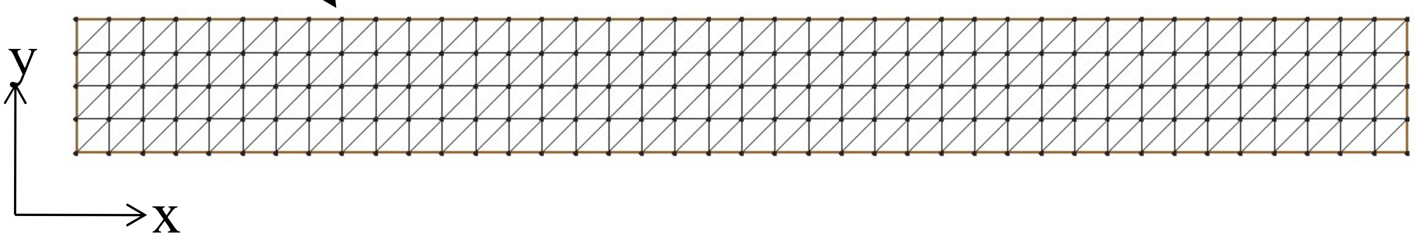
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Horizontal view



Model setup:

$\eta = \eta_0 \cos(kx) \quad k = \pi / L$

$u = w = 0$

$dx = 0.25(m)$

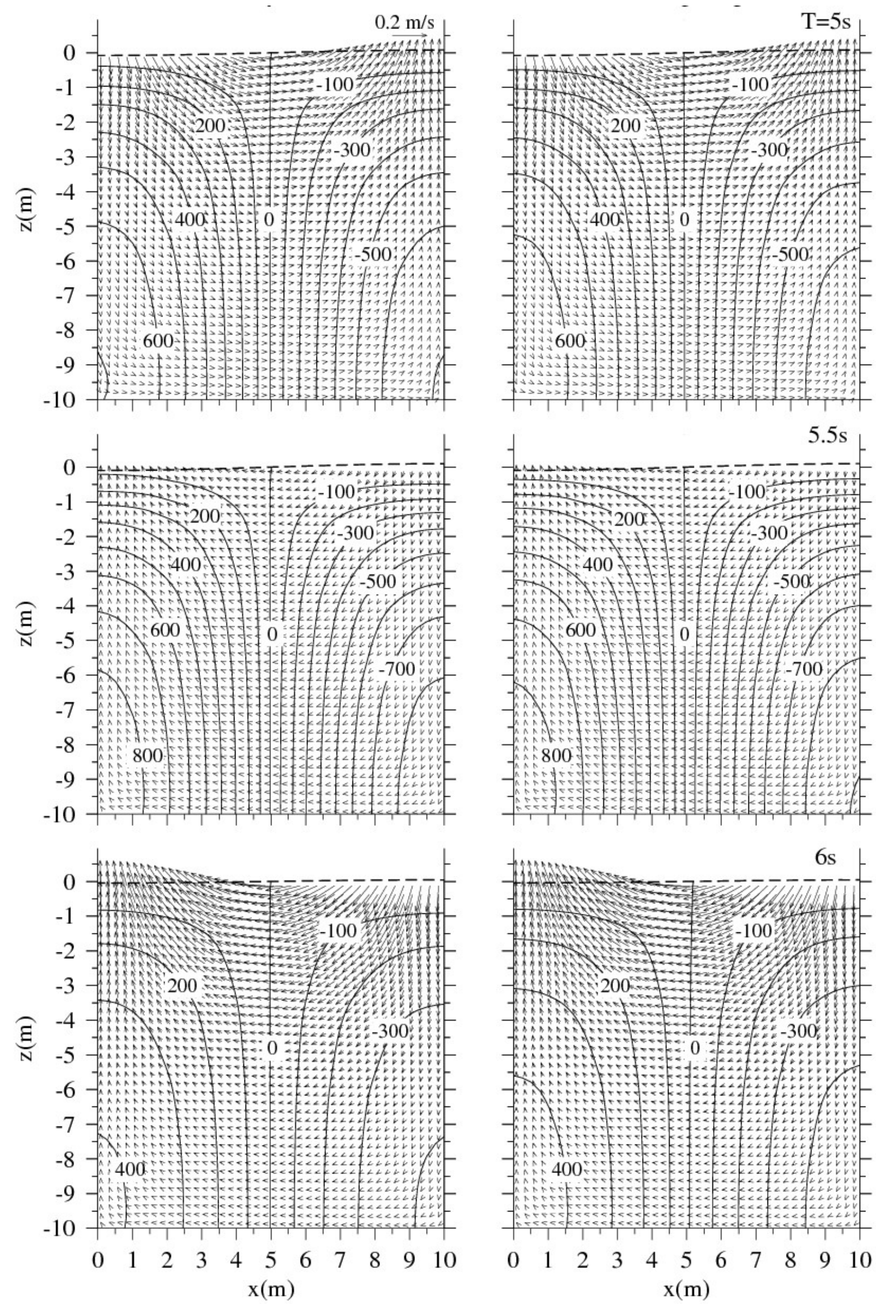
MAR513-Lec.8: Non-Hydrostatic Dynamics

Compare analytical solution (**left panel**) with numerical results (**right panel**):

•velocity field (vectors)

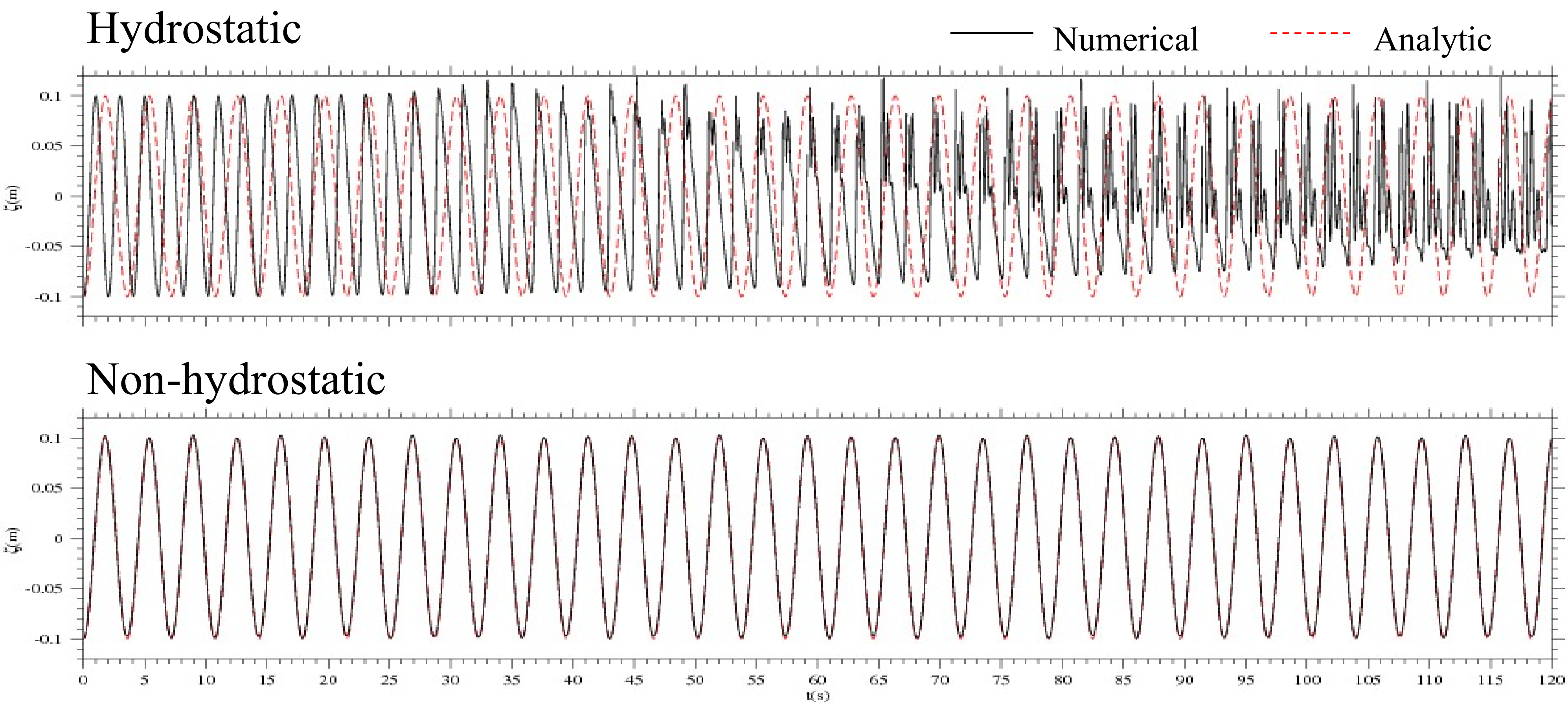
•free surface (dash line)

•non-hydrostatic pressure (contour lines)



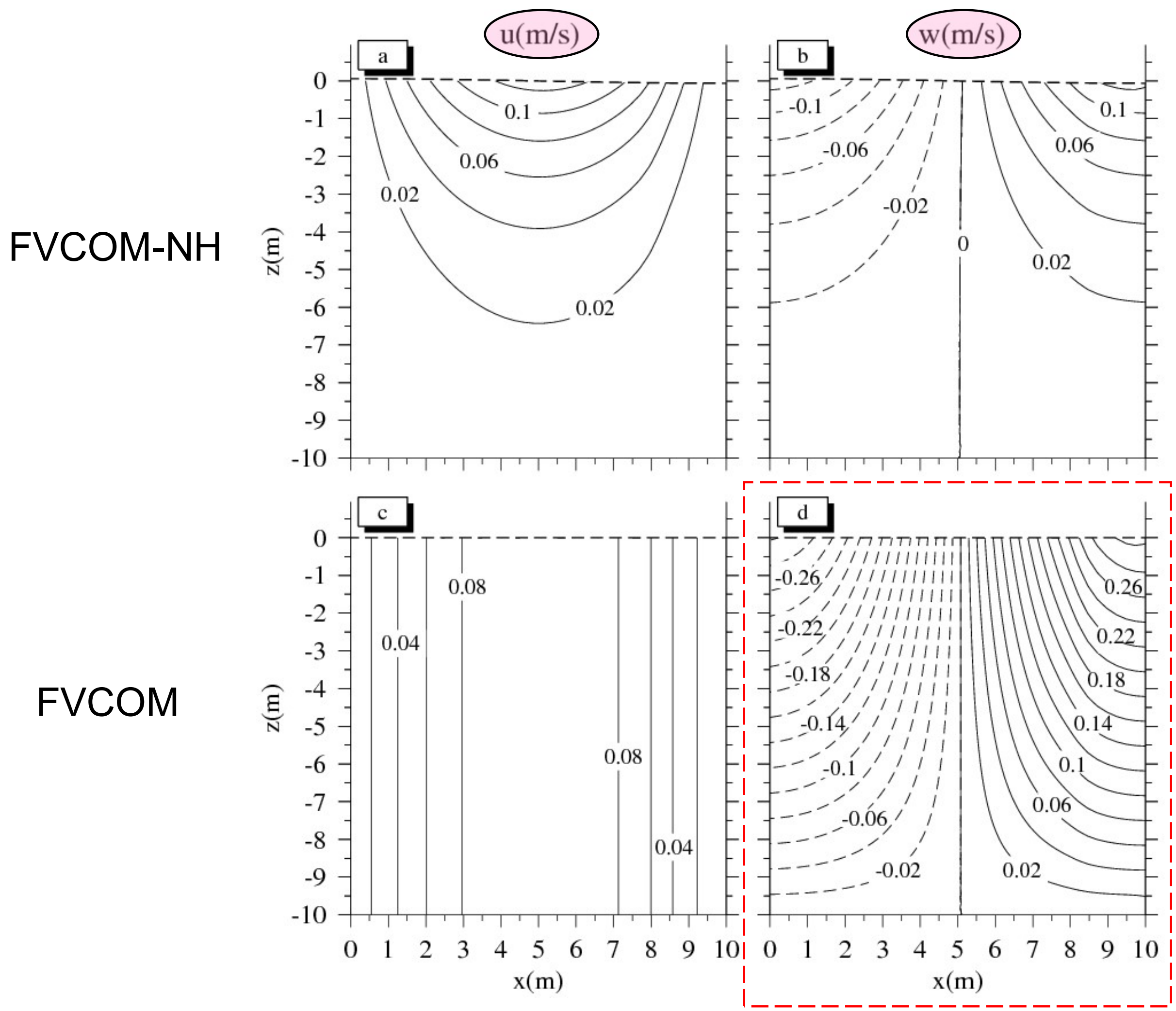
MAR513-Lec.8: Non-Hydrostatic Dynamics

Integrating the model over 600 seconds (roughly 160 wave periods) under inviscid conditions to test numerical dissipation.



MAR513-Lec.8: Non-Hydrostatic Dynamics

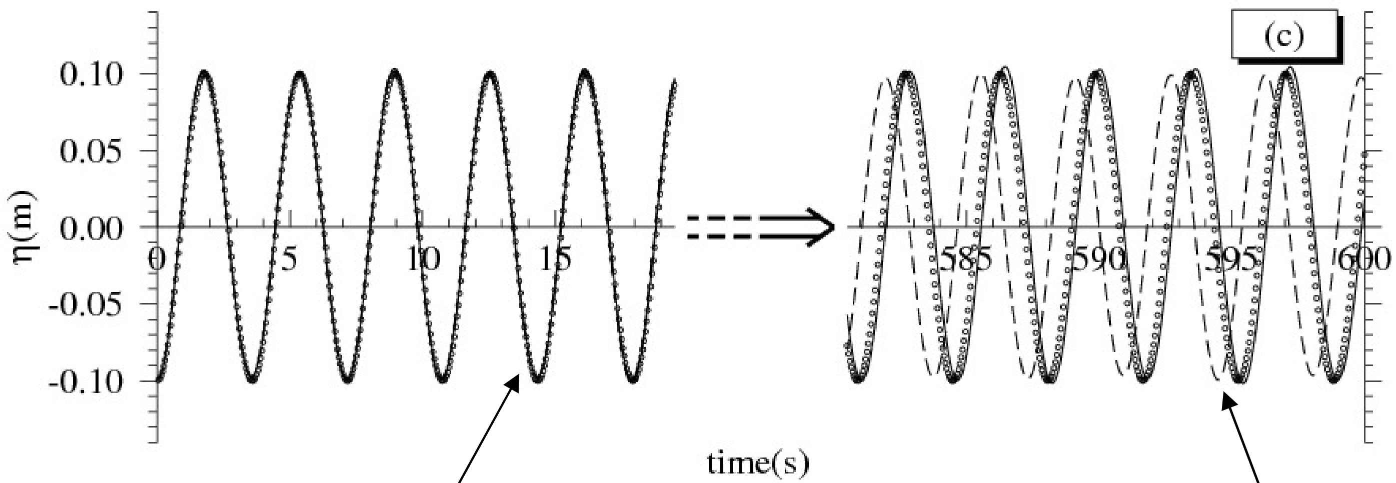
The dynamical reason why we see the hydrostatic run is not correct:

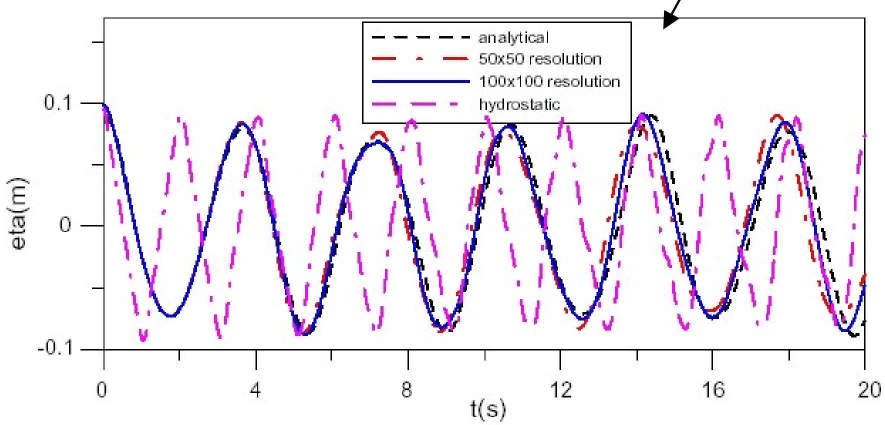


MAR513-Lec.8: Non-Hydrostatic Dynamics

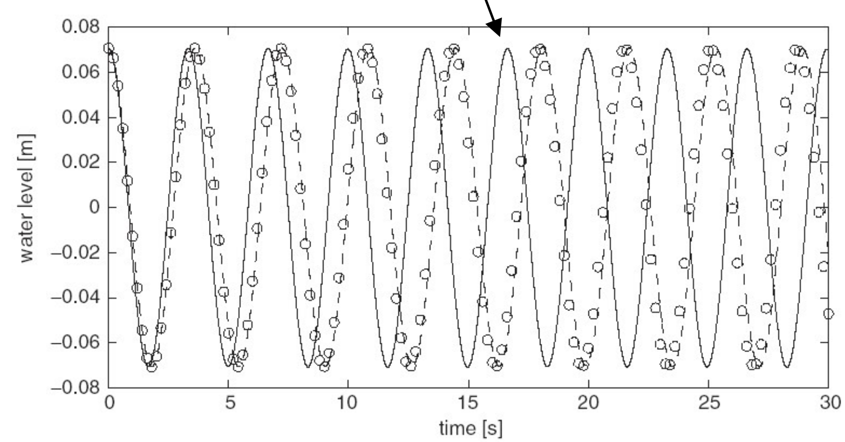
The comparison of FVCOM-NH numerical solution with other models

FVCOM-NH





Non-hydrostatic ROMs (Kanarska, 2007)

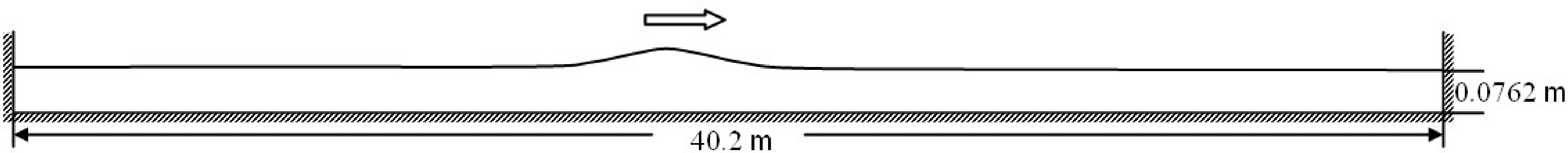


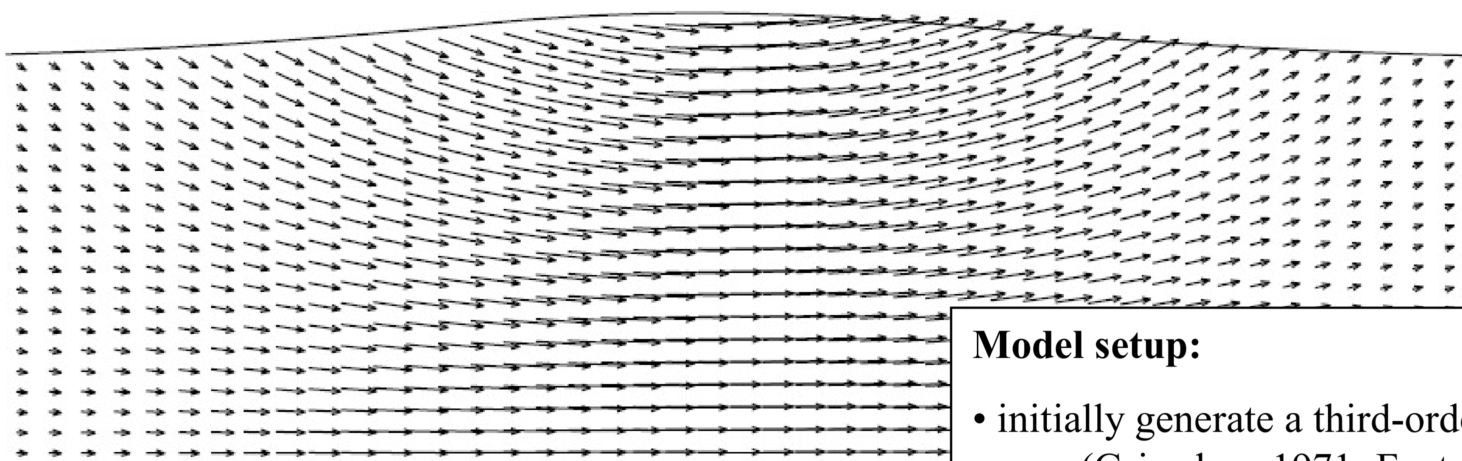
Ziljema and Stelling's (2005) test to show the free surface error that is related to Casulli's method of setting non-hydrostatic pressure to be zero in the whole first layer cell.

MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case2: surface solitary wave

(1) Over flat bottom





Model setup:

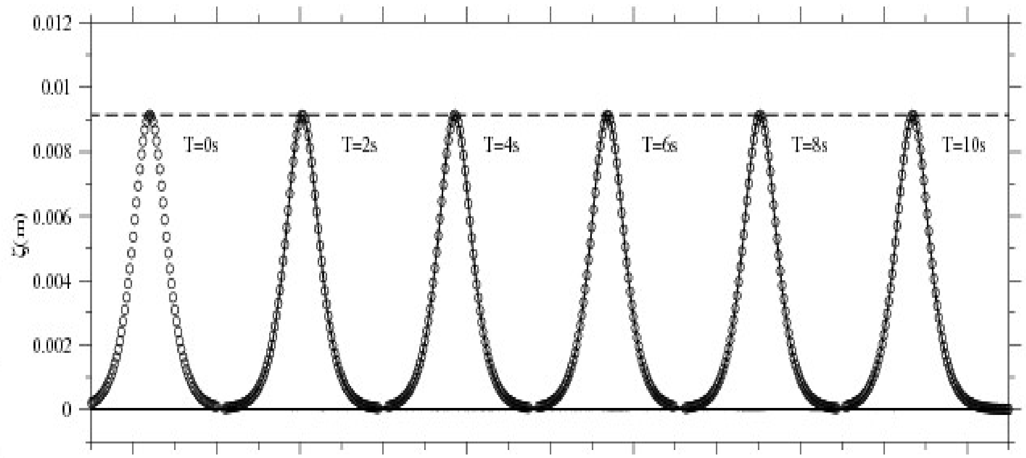
- initially generate a third-order solution of solitary wave (Grimshaw,1971; Fenton, 1972)
- h/H = 0.12
- effective wave length, L=1.6 m
- dx = 0.01(m)
- no bottom friction and eddy viscosity

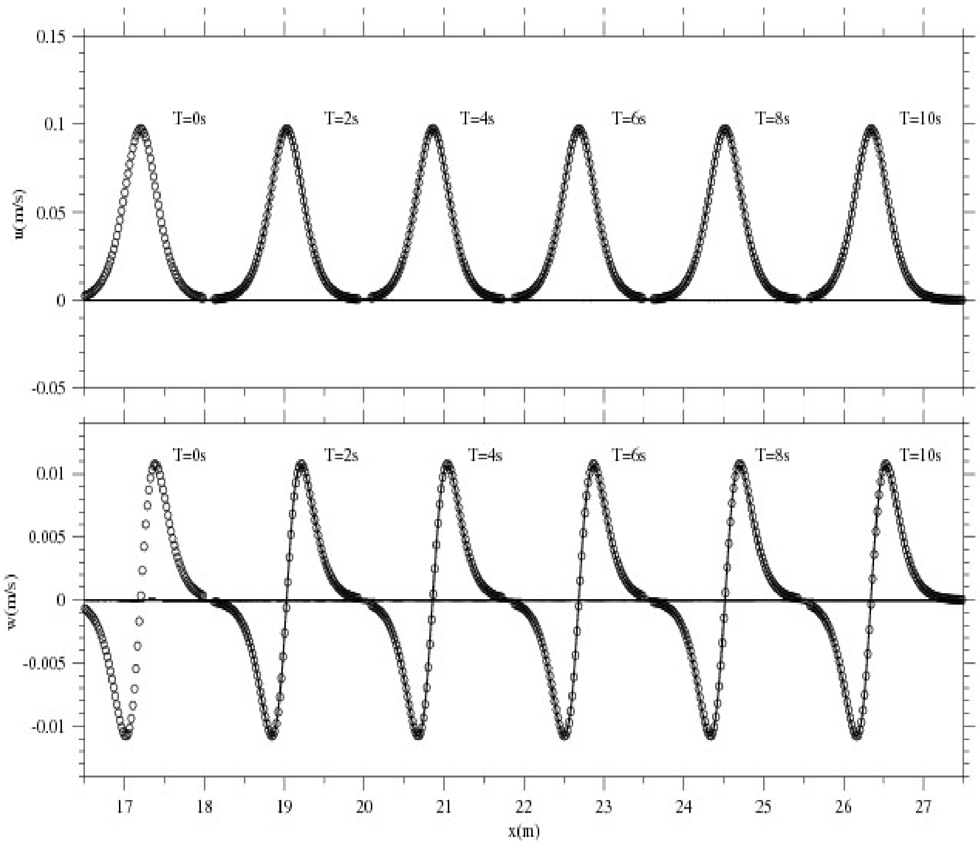
MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case2: surface solitary wave

(1) Over flat bottom

compare free surface with analytical solution

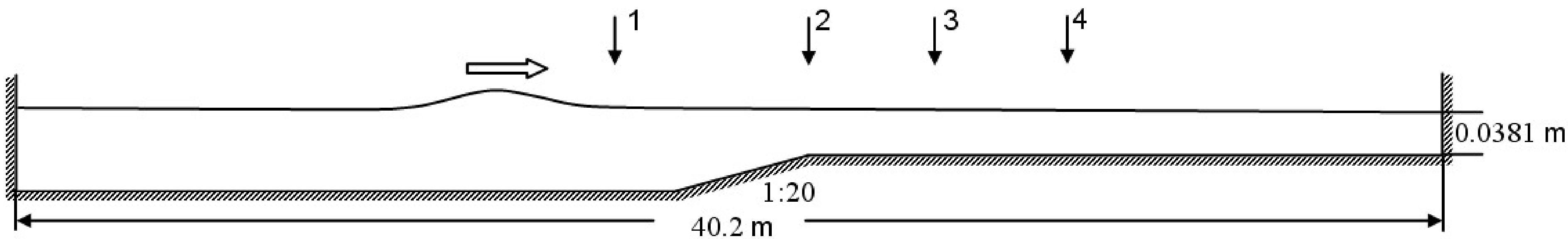




compare u and w velocity at mid-depth

Test case2: surface solitary wave

(2) Over a linear slope



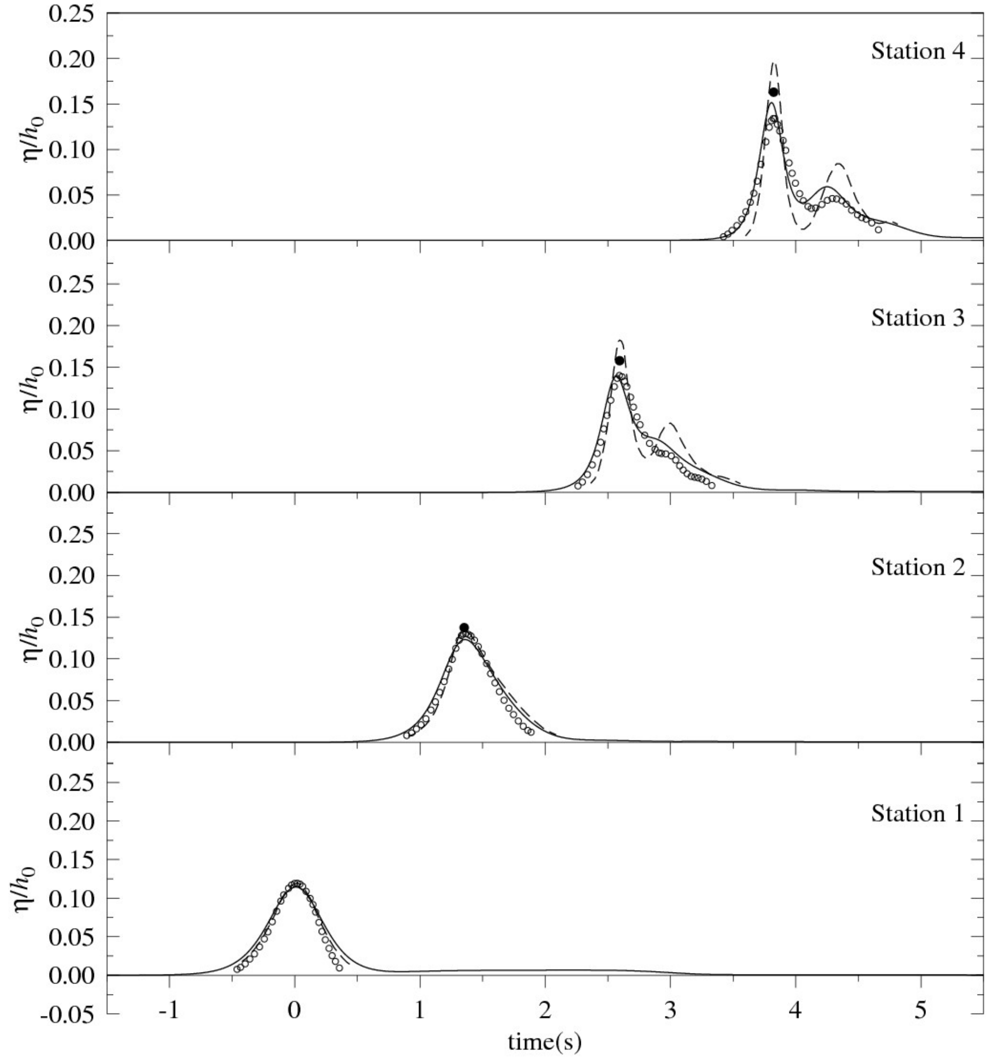
- The model setup is same as before;
- Without wave breaking, laboratory observations indicate the initial solitary wave is transformed into a train of solitary waves after it enters the shallow region, called “fission phenomena”.

MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case2: surface solitary wave

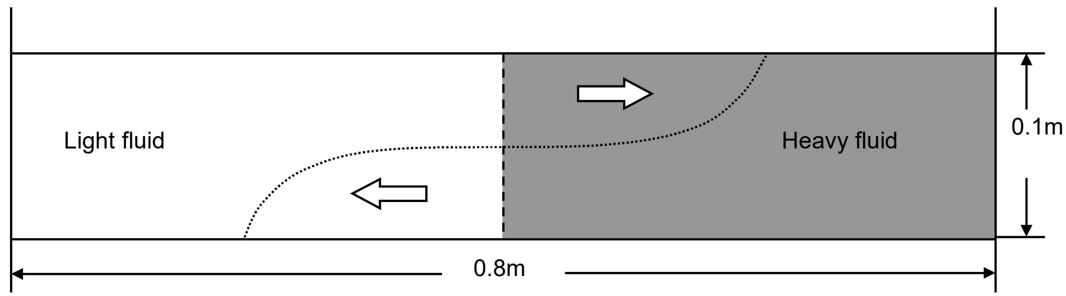
(2) Over a linear slope

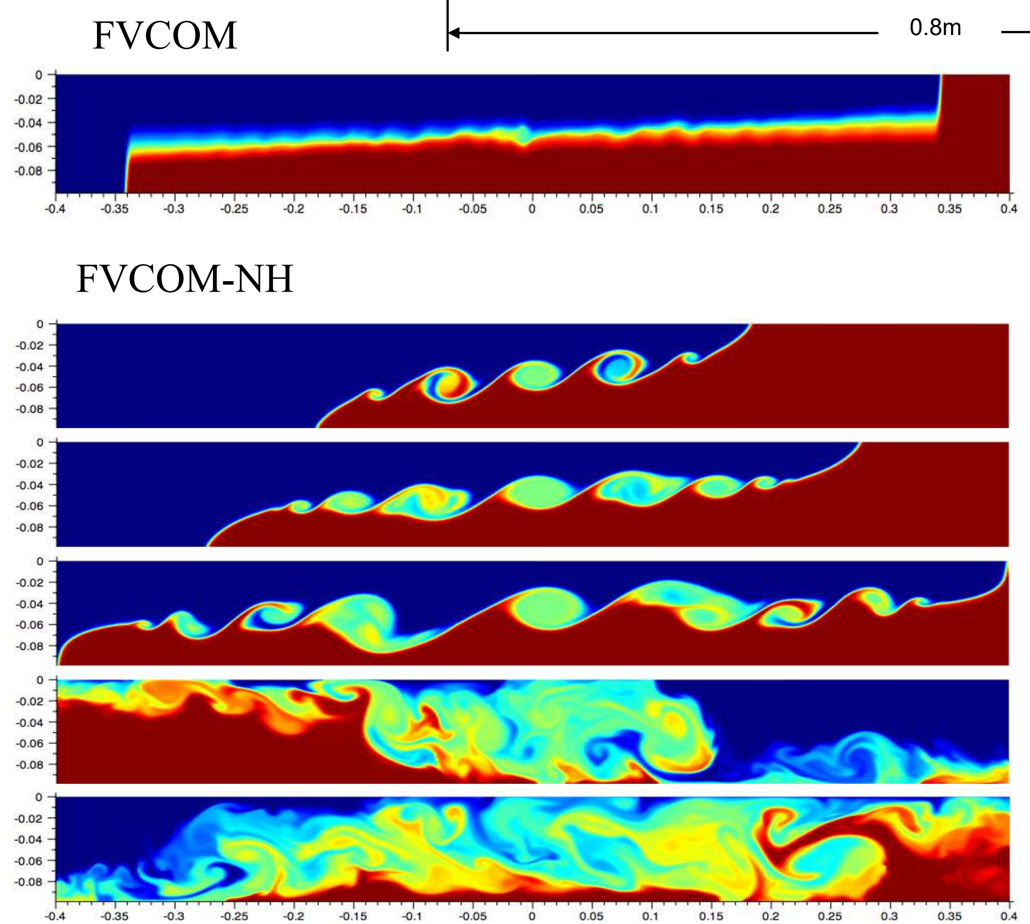
The simulated free surface variations match well with the observed at all probe stations. The fission phenomena is also well reproduced!



MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case3: lock-exchange flow





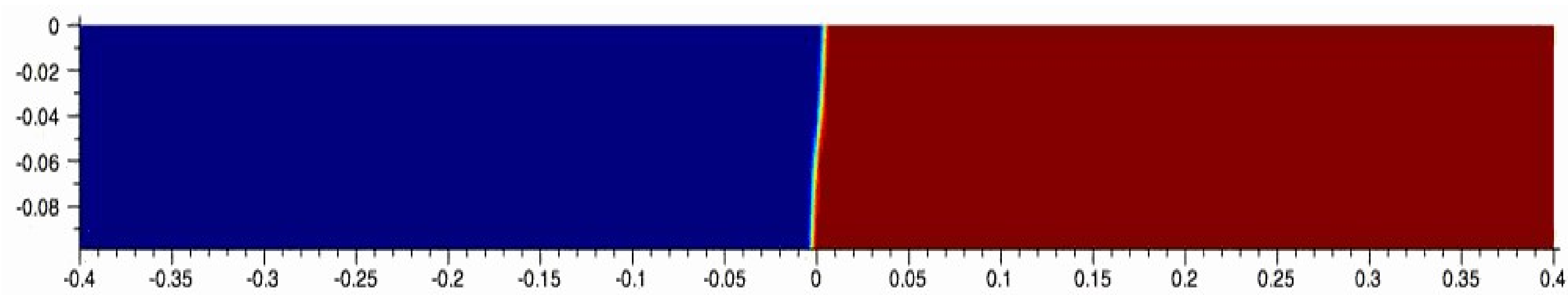
Model setup:

- Initially $g' = g\Delta\rho / \rho_0 = 0.01 m/s^2$
- Vertical 100 sigma layers
- Horizontal 400 × 5 nodes, dx = 0.002(m)
- no bottom friction and viscosity/diffusivity

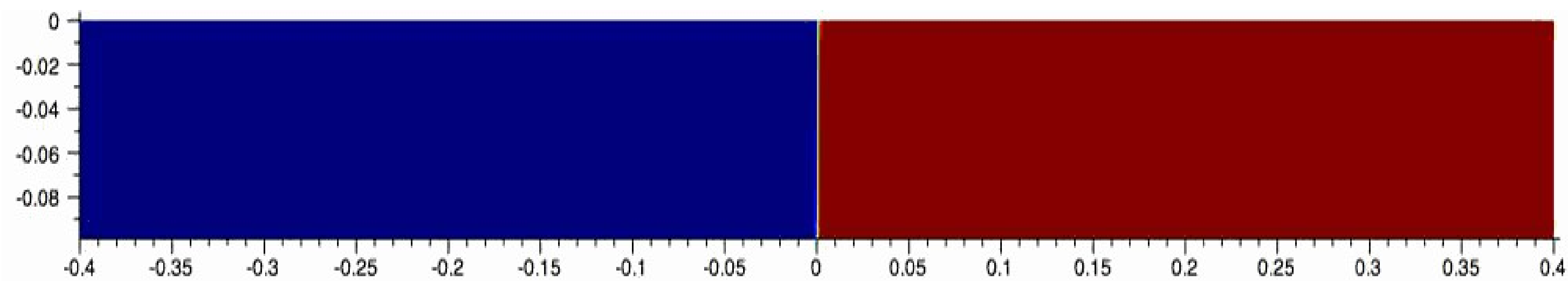
MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case3: Lock-exchange flow

Hydrostatic FVCOM



FVCOM-NH



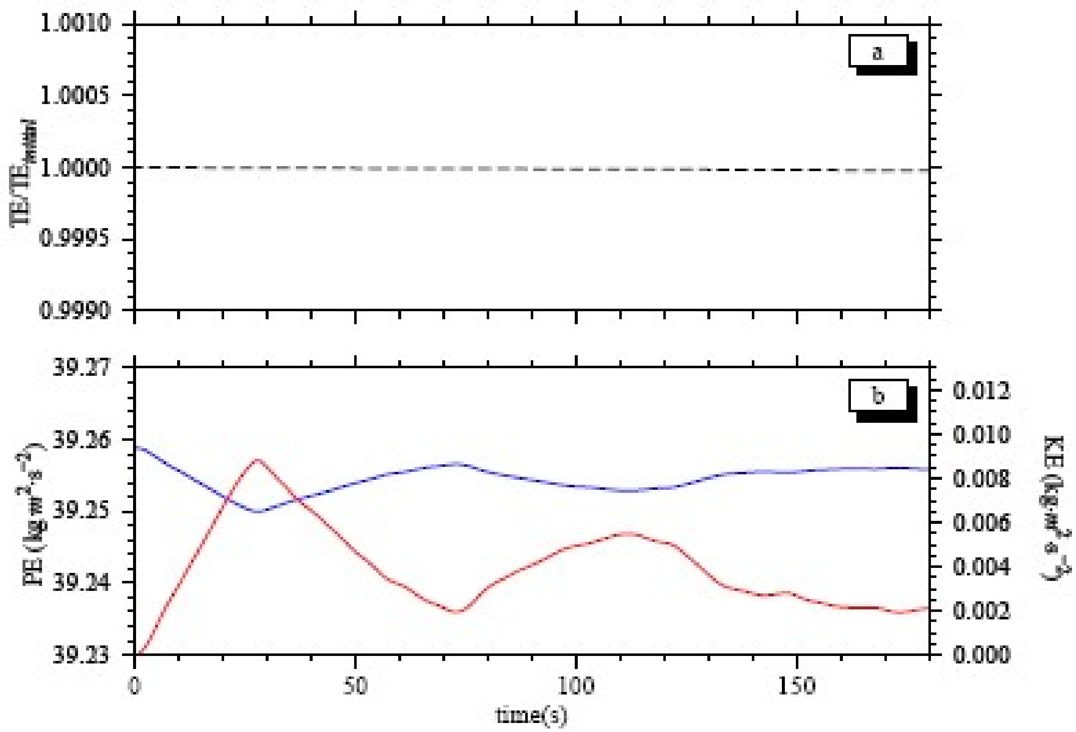
MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case3: lock-exchange flow

Define: Potential Energy = $\int_{-L/2}^{L/2}\int_{0}^{H} \rho g z dx dz$

Kinetic Energy = $\int_{-L/2}^{L/2}\int_{0}^{H} \frac{1}{2}\rho V(u^2+v^2) dx dz$

Under inviscid condition, the simulation showed a nice potential and kinetic energy transferring process and conserve the total energy to the order of 10^{-4}.



Blue line: PE

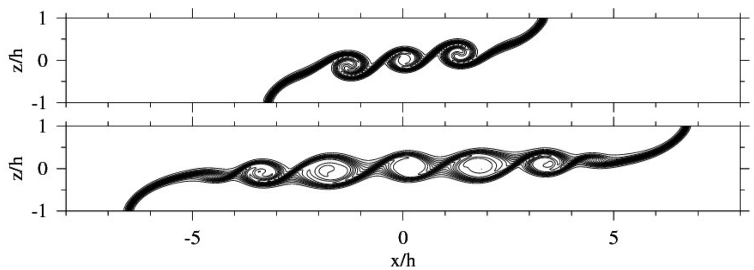
Red line: KE

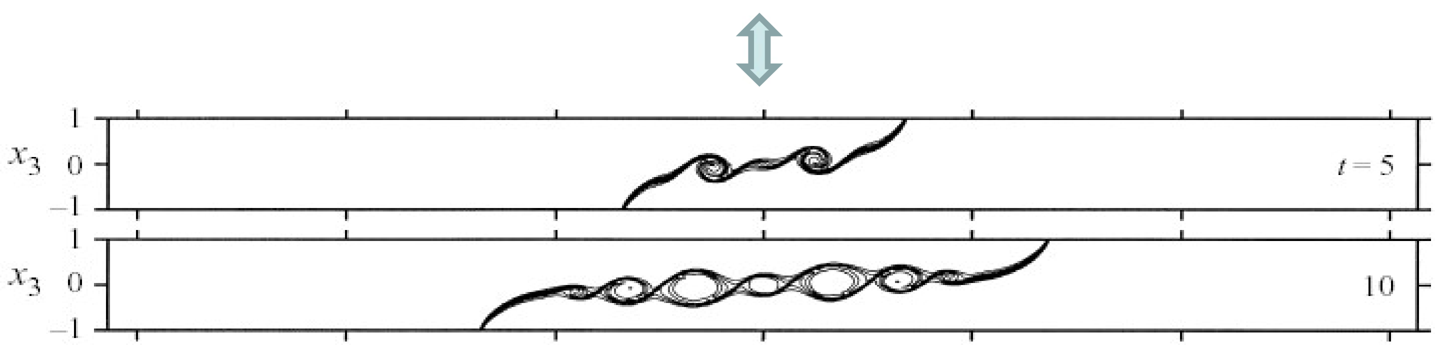
MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case3: lock-exchange flow

Comparison of FVCOM-NH result with the one from a high-order direct numerical simulation (DNS) method, with constant horizontal and vertical eddy viscosity and tracer diffusivity 1×10^{-6} m^2/s:

FVCOM-NH





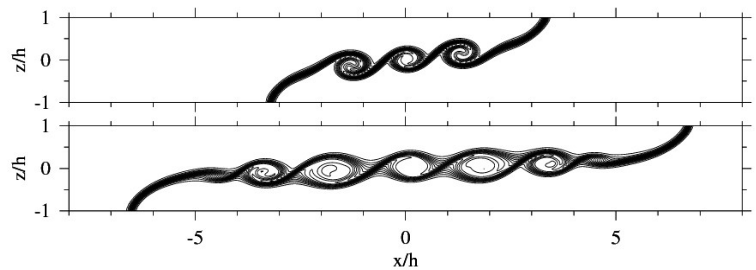
(DNS results from Härtel et al., 2000)

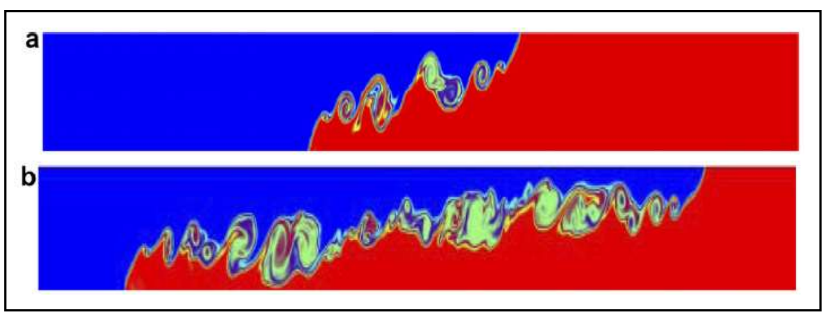
MAR513-Lec.8: Non-Hydrostatic Dynamics

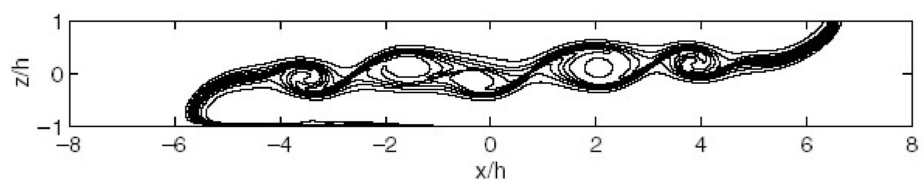
Test case3: lock-exchange flow

The comparison of FVCOM-NH numerical solution with other models

FVCOM-NH





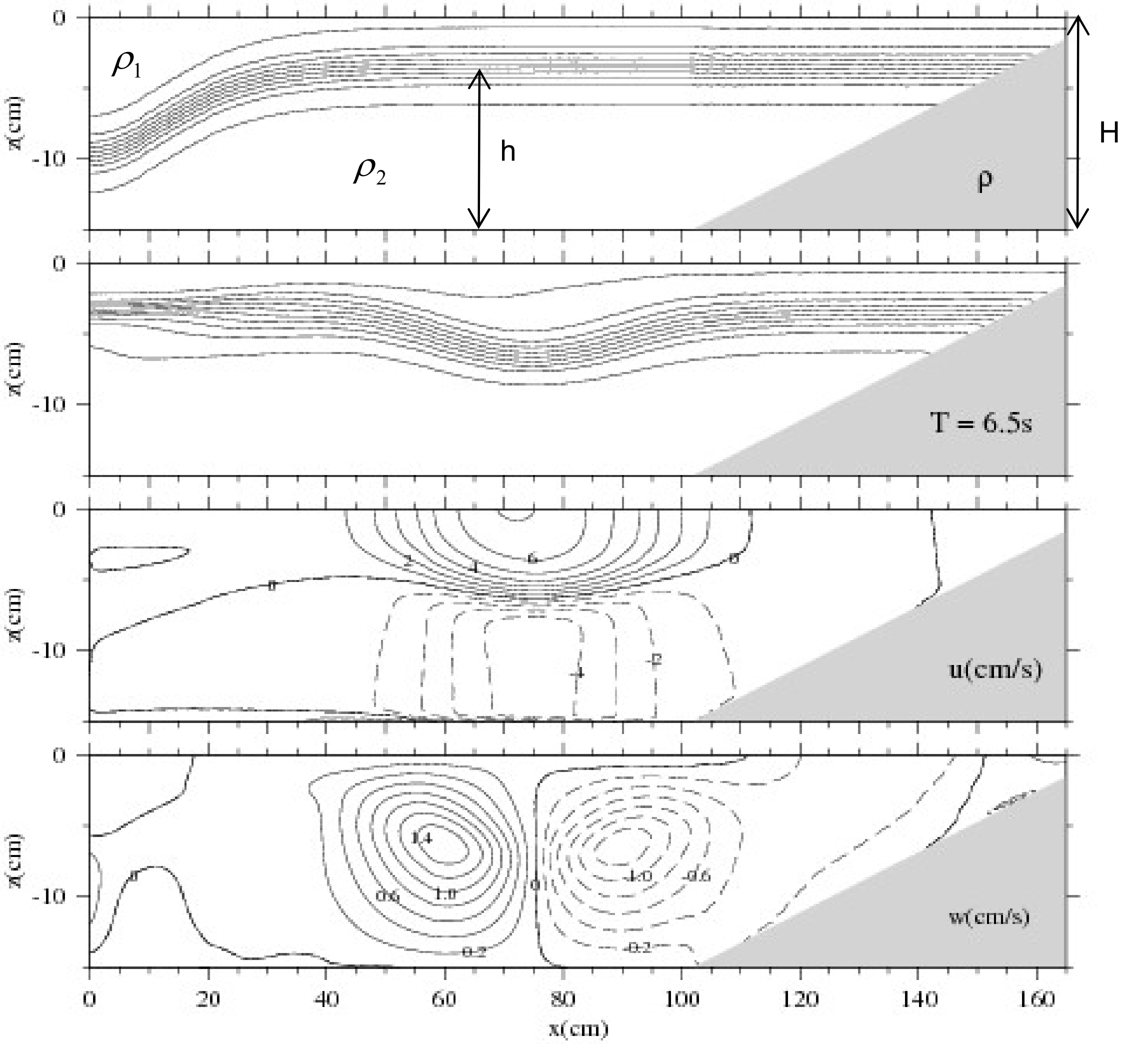


The lock-exchange problem did by Non-hydrostatic ROMS (Kanarska et al., 2007) could not show the symmetric eddies in this idealized problem.

Fringer et al. (2006) repeated this case with a similar setup but applying first-order scheme. His results was diffusive.

MAR513-Lec.8: Non-Hydrostatic Dynamics

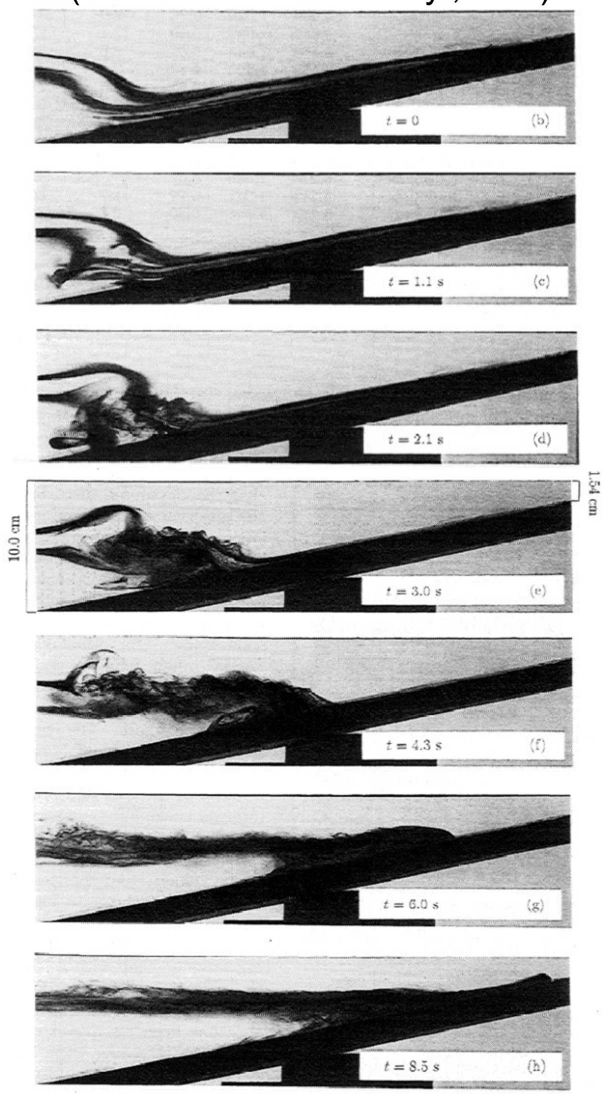
Test case4: Internal solitary waves breaking on a linear varying slope



MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case4: Internal solitary waves breaking on a linear varying slope

Photography result of exp15
(from Michallet and Ivey ,1999)



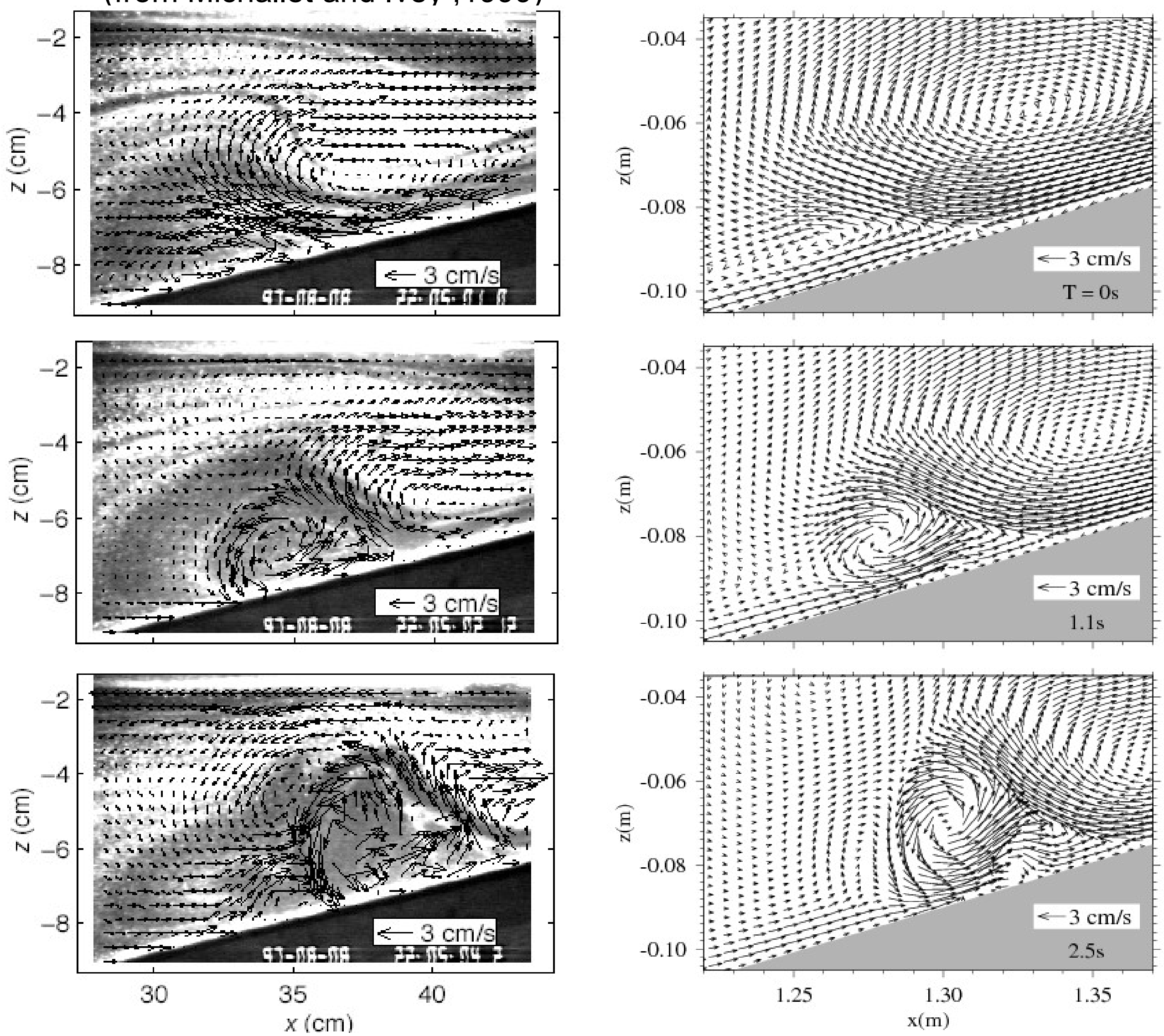
FVCOM-NH

MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case4: Internal solitary waves breaking on a linear varying slope

Photography result of exp12
(from Michallet and Ivey ,1999)

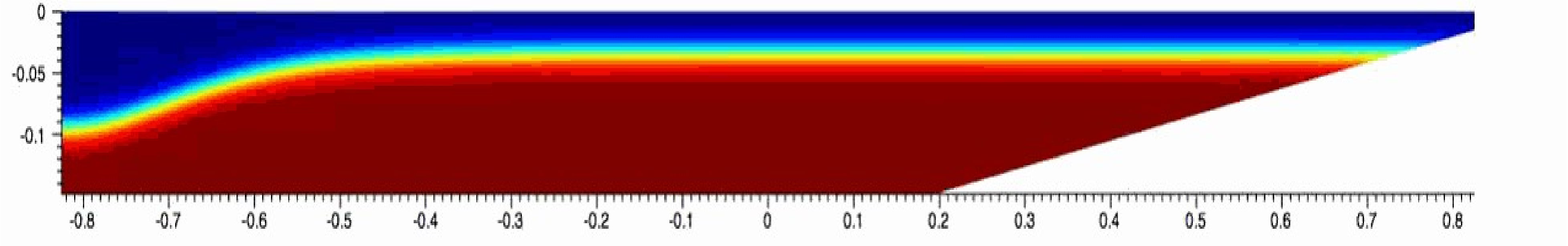
FVCOM-NH



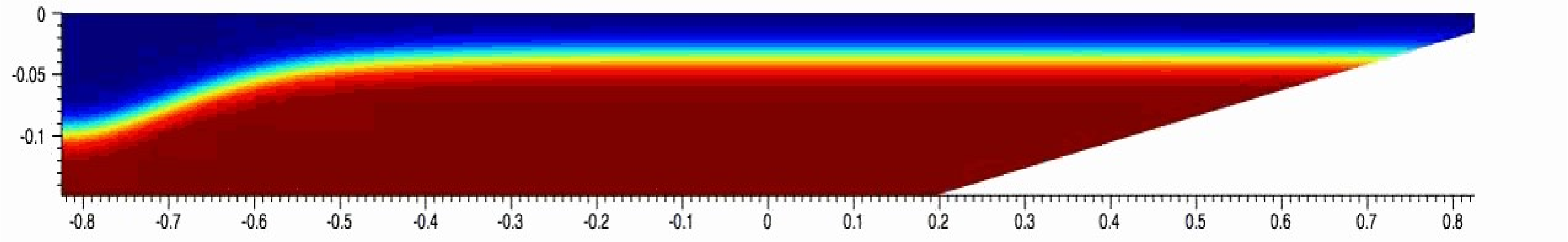
MAR513-Lec.8: Non-Hydrostatic Dynamics

Test case4: Internal solitary waves breaking on a linear varying slope

Hydrostatic FVCOM



FVCOM-NH without bottom friction and eddy viscosity/tracer diffusivity



FVCOM-NH with constant bottom friction and eddy viscosity/tracer diffusivity