MAR 513-Lec. 3: Finite-Difference Methods for Multi-Variable Equations

3.1. One-dimensional gravity waves

$$\begin{cases} \dfrac{\partial u}{\partial t} = -g\dfrac{\partial \zeta}{\partial x} \\ \dfrac{\partial \zeta}{\partial t} = -H\dfrac{\partial u}{\partial x} \end{cases} \tag{3.1}$$

where u is the x component of the velocity, ζ is the free surface (the surface elevation), g is the gravitational constant, and H is the mean water depth. Here, we assume that H is a constant.

Two independent variables: u and ζ

The solution for (3.1) is given as

$$u(x,t) = \mathrm{Re}[\hat{u}e^{i(kx-\omega t)}]; \quad \zeta(x,t) = \mathrm{Re}[\hat{\zeta}e^{i(kx-\omega t)}] \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$\omega\hat{u} = gk\hat{\zeta}; \quad \omega\hat{\zeta} = Hk\hat{u} \tag{3.3}$$

Therefore, we have derived the frequency equation as

$$\omega^2 = gHk^2 \tag{3.4}$$

And then

$$\boxed{c = \frac{\omega}{k} = \pm\sqrt{gH}} \tag{3.5}$$

The gravity wave propagates along the x axis in both directions at a speed $\sqrt{gH}$. This phase speed is not a function of wave number so the gravity wave is a non-dispersive wave.

Now, let us solve (3.1) using a finite-difference method (central difference) shown below

$$\frac{\partial u_j}{\partial t} = -g\frac{\zeta_{j+1} - \zeta_{j-1}}{2\Delta x}; \frac{\partial \zeta_j}{\partial t} = -H\frac{u_{j+1} - u_{j-1}}{2\Delta x} \tag{3.6}$$

Assume that the solutions for (3.6) is given as

$$u(x,t) = \mathrm{Re}[\hat{u}e^{i(kj\Delta x - \omega t)}]; \quad \zeta(x,t) = \mathrm{Re}[\hat{\zeta}e^{i(kj\Delta x - \omega t)}] \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\omega \hat{u} = gk \frac{\sin k\Delta x}{k\Delta x} \hat{\zeta}; \quad \omega \hat{\zeta} = Hk \frac{\sin k\Delta x}{k\Delta x} \hat{u} \tag{3.8}$$

Solving (3.8) for ω, we get

$$\omega^2 = gH \left(\frac{\sin k\Delta x}{k\Delta x} \right)^2 k^2 \tag{3.9}$$

And then,

$$c^* = \pm\sqrt{gH} \left(\frac{\sin k\Delta x}{k\Delta x} \right) = c \frac{\sin k\Delta x}{k\Delta x} \tag{3.10}$$

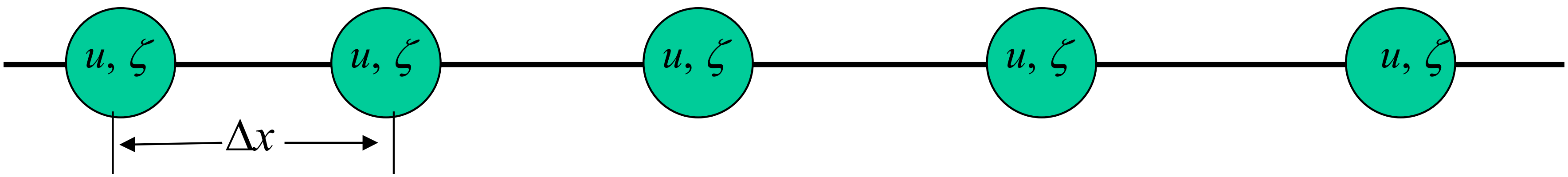
The key point is that the phase speed in the finite-difference equations is a function of wave number, so the differencing again results in computational dispersion!

The errors become larger as $k\Delta x$ increases. When $k\Delta x > \pi/2$, the numerical phase speed is opposite to the true phase speed. When $k\Delta x = \pi$, the numerical phase speed equals to zero---stationary!

The difference between this problem and the advective problem discussed before is that it includes two independent variables. Even though in this one dimensional case, there are more flexible rooms for the design of grids.

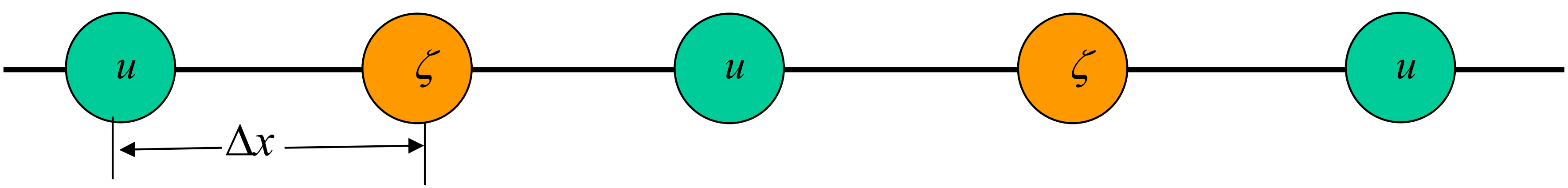
Example 1: All variables at every grid point

$$\frac{\partial u_j}{\partial t} = -g\frac{\zeta_{j+1} - \zeta_{j-1}}{2\Delta x}; \frac{\partial \zeta_j}{\partial t} = -H\frac{u_{j+1} - u_{j-1}}{2\Delta x}$$



This approach is very similar to those used in the advective equation.

Example 2: Staggered grids

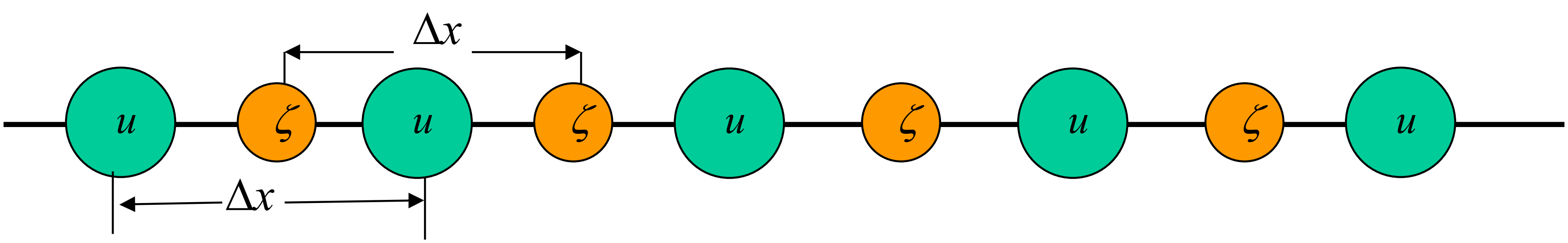


Two variables are computed at an alternate grids points

Advantage of the staggered grid:

1. Computational efficiency-reduce the computational time by a factor of two;
2. Easy boundary condition-specify the only one variable at the boundary.
3. Limitation of $2\Delta x$ jumps to $4\Delta x$

The staggered grid can also be made as the "subgrid" approach



1. Remain the same computational times as a grid that is not staggered

2. Improve the numerical accuracy.

3.2. Two-dimensional gravity waves

$$\begin{cases} \frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x}; \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial t} = -H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \end{cases} \tag{3.11}$$

Substituting the wave solutions as

$$u(x, y, t) = \mathrm{Re}[\hat{u}e^{i(kx+ly-\omega t)}]; v(x, y, t) = \mathrm{Re}[\hat{v}e^{i(kx+ly-\omega t)}] \quad \zeta(x, y, t) = \mathrm{Re}[\hat{\zeta}e^{i(kx+ly-\omega t)}] \tag{3.12}$$

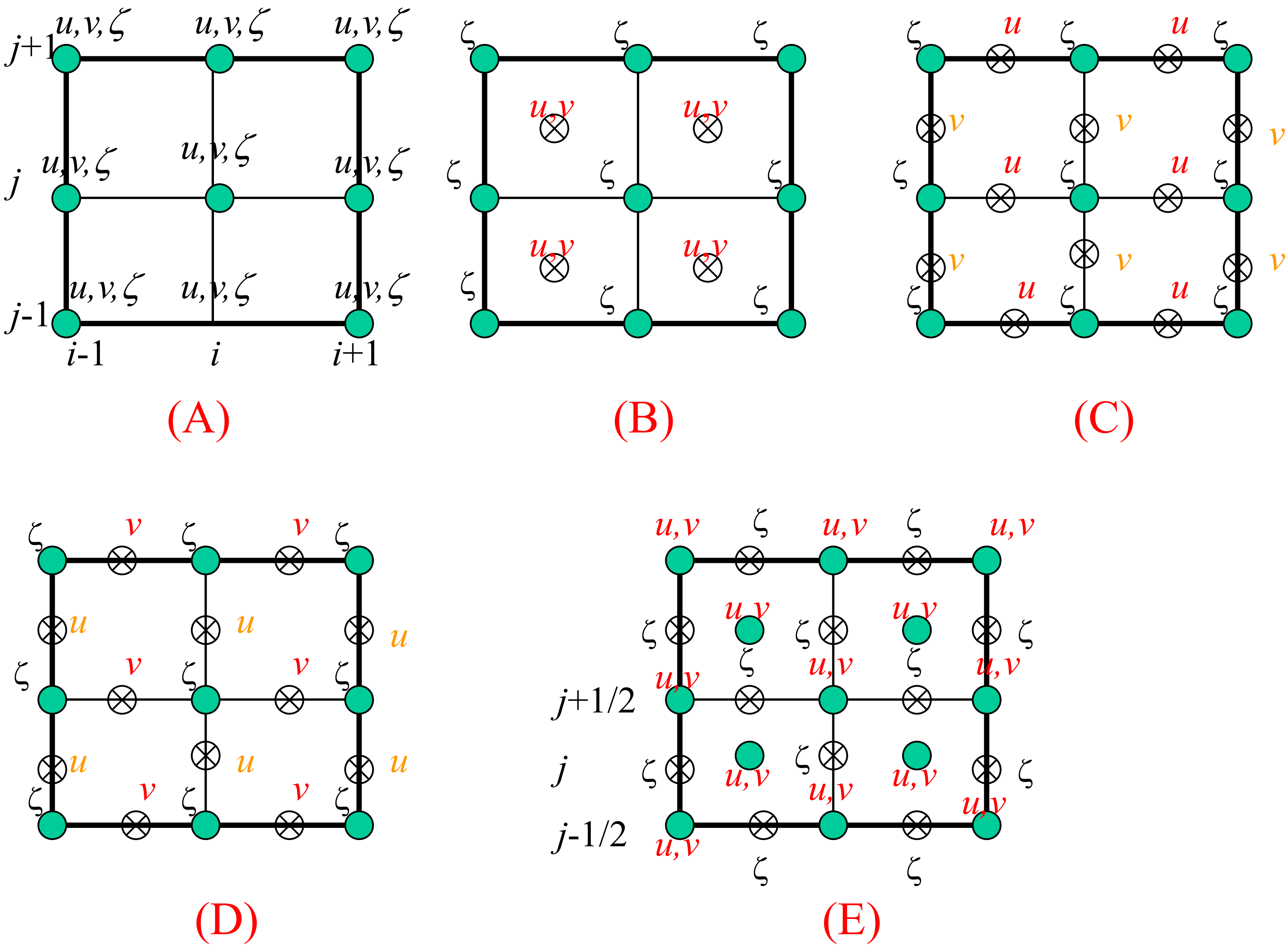
we have

$$\omega^2 = gH(k^2 + l^2) \tag{3.13}$$

And then

$$c = \frac{\omega}{\sqrt{k^2 + l^2}} = \pm\sqrt{gH} \tag{3.14}$$

Now, we can have a large number of spatial arrangement of the variables (Arakawa, 1972)



Assume that a uniform grid is used for both x and y directions. That is

$$\Delta x = \Delta y = d$$

, and uses the central difference schemes for x and y gradient terms. We get

$$\delta_x \zeta = \frac{1}{2d}[\zeta(x+d,y) - \zeta(x-d,y)]; \delta_y \zeta = \frac{1}{2d}[\zeta(x,y+d) - \zeta(x,y-d)] \qquad (3.15)$$

Therefore, the difference equations for (3.11) can be rewritten as

$$\frac{\partial u}{\partial t} = -g\delta_x \zeta; \quad \frac{\partial v}{\partial t} = -g\delta_y \zeta; \quad \frac{\partial \zeta}{\partial t} = -H(\delta_x u + \delta_y v); \qquad (3.16)$$

Substituting the solution of

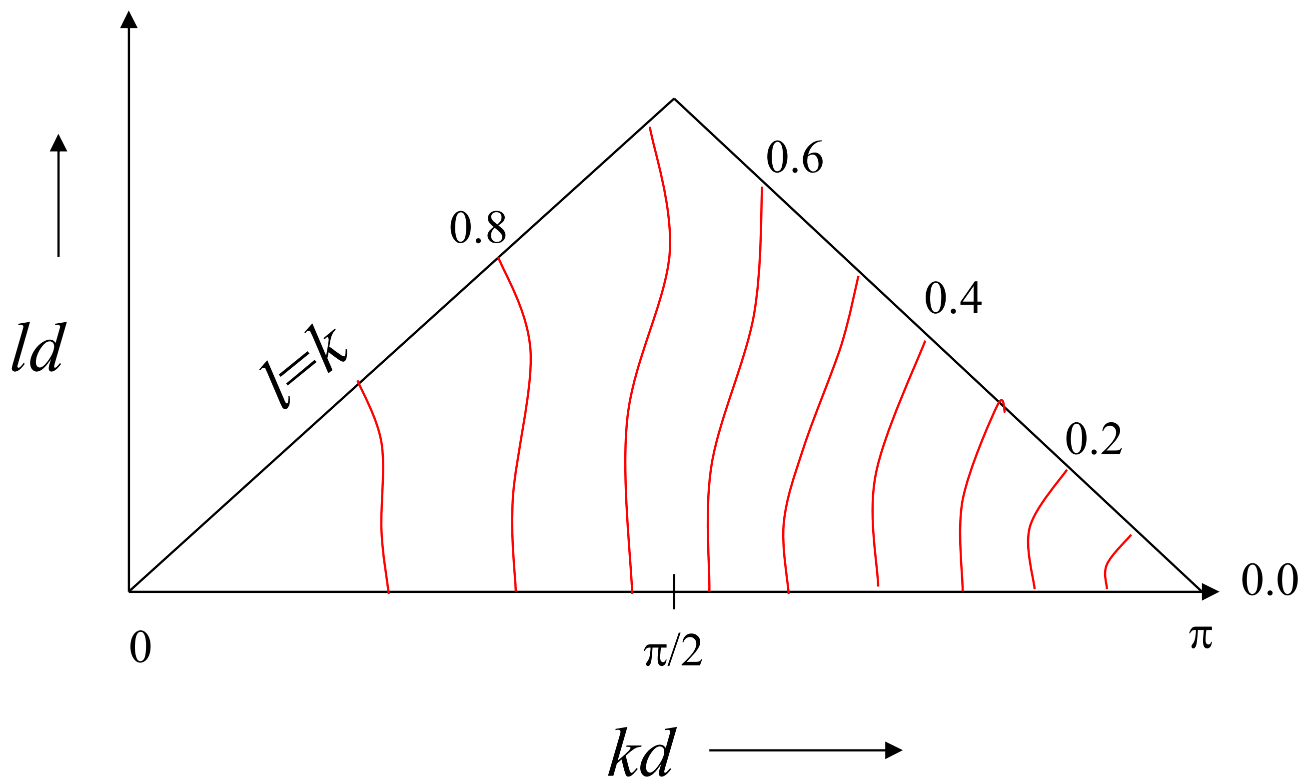
$$u(x,y,t) = \mathrm{Re}[\hat{u}e^{i(kx+ly-\omega t)}]; v(x,y,t) = \mathrm{Re}[\hat{v}e^{i(kx+ly-\omega t)}] \quad \zeta(x,y,t) = \mathrm{Re}[\hat{\zeta}e^{i(kx+ly-\omega t)}] \qquad (3.16)$$

We have the frequency equation

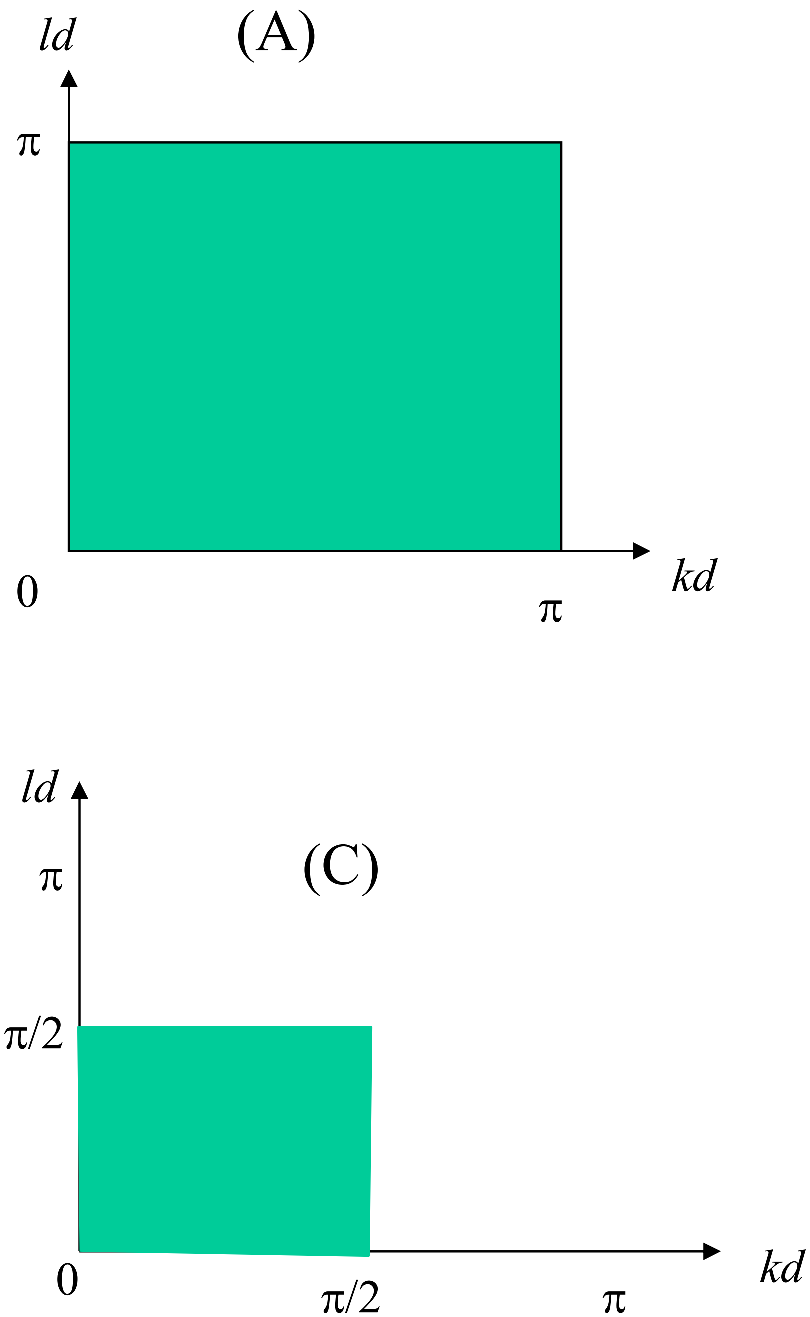
$$\omega^2 = gH\frac{\sin^2 kd + \sin^2 ld}{d^2} \longrightarrow c^* = \pm\sqrt{gH}\sqrt{\frac{\sin^2 kd + \sin^2 ld}{(kd)^2 + (ld)^2}} \qquad (3.17)$$

The ratio of numerical phase speed to the true phase speed is

$$\frac{c}{\sqrt{gH}} = \sqrt{\frac{\sin^2 kd + \sin^2 ld}{(kd)^2 + (ld)^2}} \tag{3.18}$$



Relative phase speed of gravity wave when the space derivative is approximated by the central difference scheme.



Admissible region of wave numbers for the lattices A, E and C

1) Grid C gives a more accurate phase speed for gravity wave than the other grids.

2) Grid E is superior to Grid A. With the same truncation error, Grid E can be achieved in about half of the computational time (exactly half if the equations are linear);

3) For

$\Delta x = \Delta y = d$ when $kd = \pi$ and $ld = \pi$, ie. $k = l$,

$$c^* = 0$$

The numerical solution represents a stationary wave: This is also called "two-grid interval wave". This type wave can easily be generated when boundary conditions are artificially described.

Grid C is not perfect grid. For example,

a) Since u and v are not at the same point, there are some difficulties with the Coriolis force terms;

b) Since the along the boundary current needs to be specified, it makes difficult to get the radiation boundary condition at an open boundary.

3.3. Gravity-inertial waves

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} + fv\,; \quad \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial y} - fu; \quad \frac{\partial \zeta}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{3.18}$$

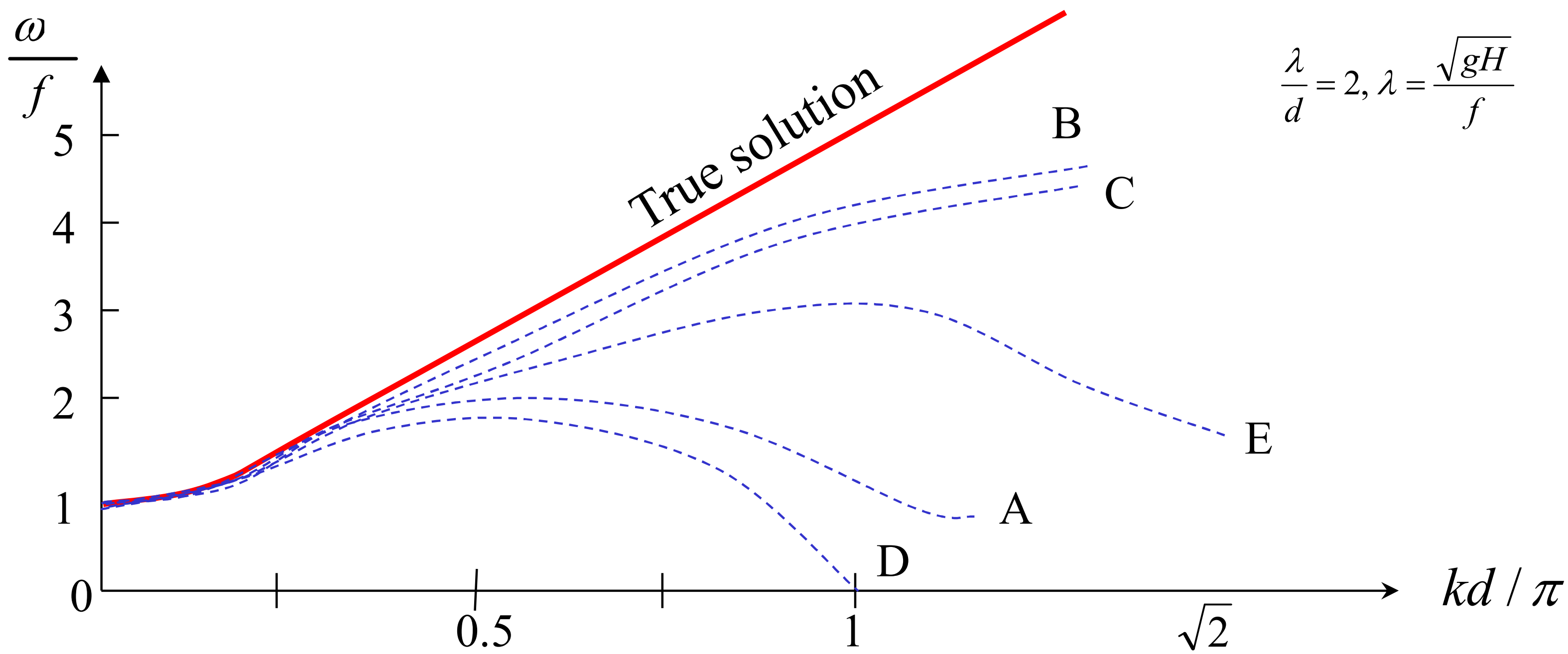
On the large scale, we consider two distinct types of motion:

a) low-frequency , quasi-geostrophic and quasi-nondivergence flow;

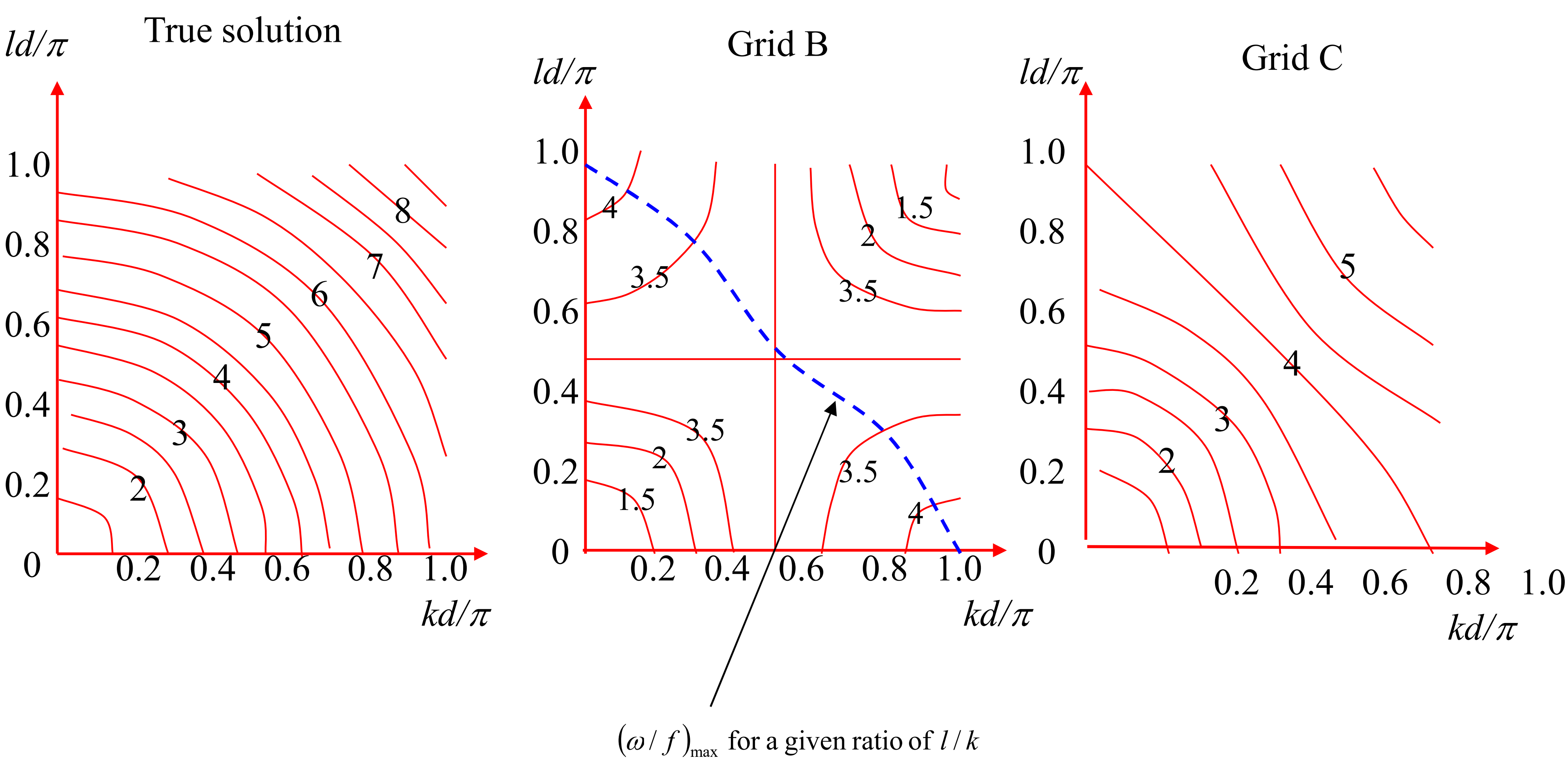
b) high-frequency gravity-inertial waves.

For the case with

$$\frac{\lambda}{d} = 2, \lambda = \frac{\sqrt{gH}}{f}$$



For the case with $\frac{\lambda}{d} = 2, \lambda = \frac{\sqrt{gH}}{f}$



Conclusion

1) Introducing the staggered grid will improve the computational efficiency and also numerical accuracy.

2) For dispersion relationship, Grid B is better than Grid C, but both are two better grids, and Grid A and Grid D are two inadequacy grids.

3) Grid B and Grid E can be converted each other by a rotation through an angle of $\pi/4$, but the results are so different!

4) As long as both dispersion and phase plan is considered, Grid C is the best choice for the gravity wave.

QS. How about open boundary conditions?

Next lecture!