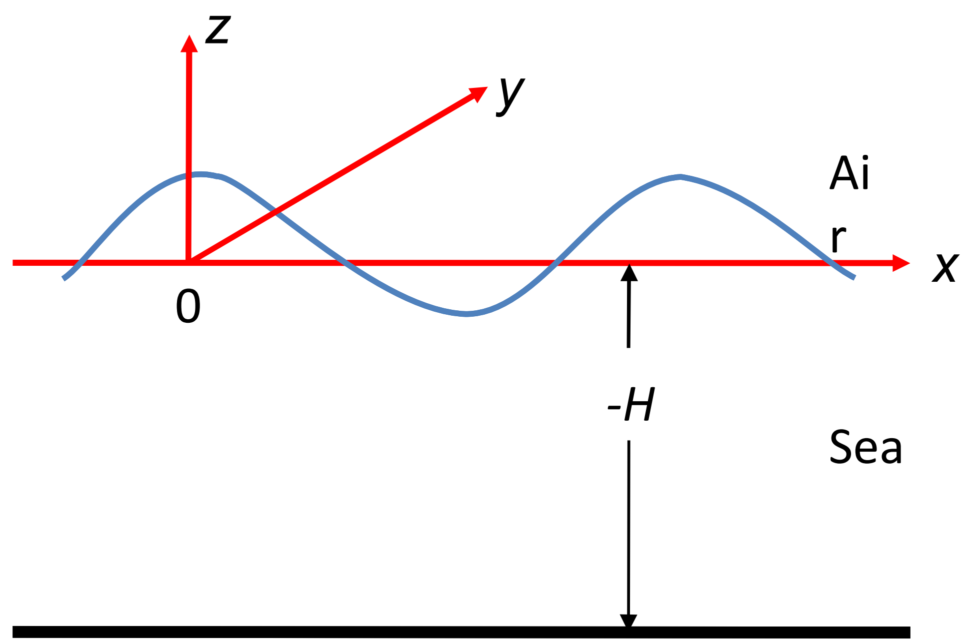
MAR513 Lecture 11: Surface Wave Modeling



Conditions:
a) no rotation; b) incompressible; c) linear

$$P = P_o + p'$$

$$u = \frac{dx}{dt}; \quad v = \frac{dy}{dt}; \quad w = \frac{dz}{dt}$$

The perturbation momentum and continuity equations can be written as

$$\begin{cases} \rho_o \dfrac{\partial u}{\partial t} = -\dfrac{\partial p'}{\partial x} \\ \rho_o \dfrac{\partial v}{\partial t} = -\dfrac{\partial p'}{\partial y} \\ \rho_o \dfrac{\partial w}{\partial t} = -\dfrac{\partial p'}{\partial z} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

Boundary condition:

At the bottom: $w\big|_{z=-H} = 0$

At the free surface: $w\big|_{z=0} = \dfrac{\partial \zeta}{\partial t}$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=\rho_o\frac{\partial}{\partial x}(\frac{\partial u}{\partial t})+\frac{\partial}{\partial y}(\frac{\partial v}{\partial t})+\frac{\partial}{\partial z}(\frac{\partial w}{\partial t})=-[\frac{\partial}{\partial x}(\frac{\partial p'}{\partial x})+\frac{\partial}{\partial y}(\frac{\partial p'}{\partial y})+\frac{\partial}{\partial z}(\frac{\partial p'}{\partial z})]=0$$

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 p'}{\partial y^2}+\frac{\partial^2 p'}{\partial z^2}=0$$

Laplace’s equation

Assume that the solution is

$$p'=A(t)e^{i(kx+ly+mz)}$$

$$A(t)(k^2+l^2+m^2)=0 \Rightarrow \text{since } A(t)\neq 0; \text{ then } k^2+l^2+m^2=0$$

Therefore,

$$k^2+l^2+m^2=0$$

This is impossible!

Instead, we will try

$$p' = A(t)e^{i(kx+ly)+mz}$$

then,

$$-(k^2 + l^2) + m^2 = 0 \Rightarrow m = \pm\sqrt{k^2 + l^2}$$

Assume that the solution is

$$p' = A(t)e^{[i(kx+ly)+mz]} + B(t)e^{[i(kx+ly)-mz]}$$

where $k^2 + l^2 = m^2$

At the bottom, $w|_{z=-H} = 0$; $\frac{\partial p'}{\partial z} = -\frac{\partial w}{\partial t} = 0$

$$mA(t)e^{[i(kx+ly)-mH]} + -B(t)me^{[i(kx+ly)+mH]} = 0 \Rightarrow B(t) = A(t)e^{-2mH} = A(t)e^{-2\sqrt{k^2+l^2}H}$$

So, the solution can be written as

$$p' = A(t)e^{i(kx+ly)}(e^{z\sqrt{k^2+l^2}} + e^{-2H\sqrt{k^2+l^2}}e^{-z\sqrt{k^2+l^2}})$$

$$= 2A(t)e^{H\sqrt{k^2+l^2}}e^{i(kx+ly)}(e^{(z+H)\sqrt{k^2+l^2}} + e^{-(H+z)\sqrt{k^2+l^2}})/2$$

$$= \overline{\mathrm{A}}(\mathrm{t})e^{i(kx+ly)}\mathbf{cosh}[\sqrt{k^2+l^2}(z+H)]$$

At the surface,

$$w\big|_{z=0} = \frac{\partial\zeta}{\partial t} \qquad\qquad p'\big|_{z=0} = \rho_o g\zeta \Rightarrow \zeta = p'\big|_{z=0}/\rho_o g$$

Therefore,

$$\zeta = \frac{\overline{\mathrm{A}}(\mathrm{t})}{g\rho_{\mathrm{o}}}e^{i(kx+ly)}\mathbf{cosh}[H\sqrt{k^2+l^2}];$$

$$w = \frac{1}{g\rho_{\mathrm{o}}}\frac{\partial\overline{\mathrm{A}}(\mathrm{t})}{\partial t}e^{i(kx+ly)}\mathbf{cosh}[H\sqrt{k^2+l^2}]$$

Also,

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} = -\frac{\overline{\mathrm{A}}(\mathrm{t})}{\rho_{\mathrm{o}}}\sqrt{k^2+l^2}\,e^{i(kx+ly)}\,\mathbf{sinh[}(z+H)\sqrt{k^2+l^2}\,\mathbf{]}$$

Find the frequency ω.   Assume that

$$\overline{A}(t) = A_o e^{-i\omega t}$$

At the free surface,

$$\left.\frac{\partial w}{\partial t}\right|_{z=0} = \frac{\partial^2 \zeta}{\partial t^2}$$

$$-\frac{\mathrm{A_o}}{\rho_{\mathrm{o}}}\sqrt{k^2+l^2}\,\mathbf{sinh(}H\sqrt{k^2+l^2}\,\mathbf{)} = -\frac{\omega^2 \mathrm{A_o}}{\rho_{\mathrm{o}} g}\,\mathbf{cosh[}H\sqrt{k^2+l^2}\,\mathbf{]}$$

$$\omega^2 = \mathrm{g}\sqrt{k^2+l^2}\,\mathbf{tanh(}H\sqrt{k^2+l^2}\,\mathbf{)}$$

The solution of the surface gravity wave:

$$p' = A_o e^{i(kx+ly-\omega t)} \cosh[(z+H)\sqrt{k^2+l^2}];$$

$$\zeta = \frac{A_o}{g\rho_o} e^{i(kx+ly-\omega t)} \cosh[H\sqrt{k^2+l^2}];$$

$$w = -\frac{i\omega A_o}{g\rho_o} e^{i(kx+ly-\omega t)} \cosh[H\sqrt{k^2+l^2}]$$

$$\omega^2 = g\sqrt{k^2+l^2}\tanh(H\sqrt{k^2+l^2})$$

Phase speed: Assume that the total wave number

$$\sqrt{k^2+l^2} = k_o;\ k_o = 2\pi/L_o,\ \text{where } L_o \text{ is the wavelength}$$

$$k_o x' - \omega t = \text{constant} \Rightarrow k_o \frac{dx'}{dt} - \omega = 0 \Rightarrow v_p = \frac{dx'}{dt} = \frac{\omega}{k_o}$$

$$v_{px} = \frac{\omega}{k}; \quad v_{py} = \frac{\omega}{l}$$

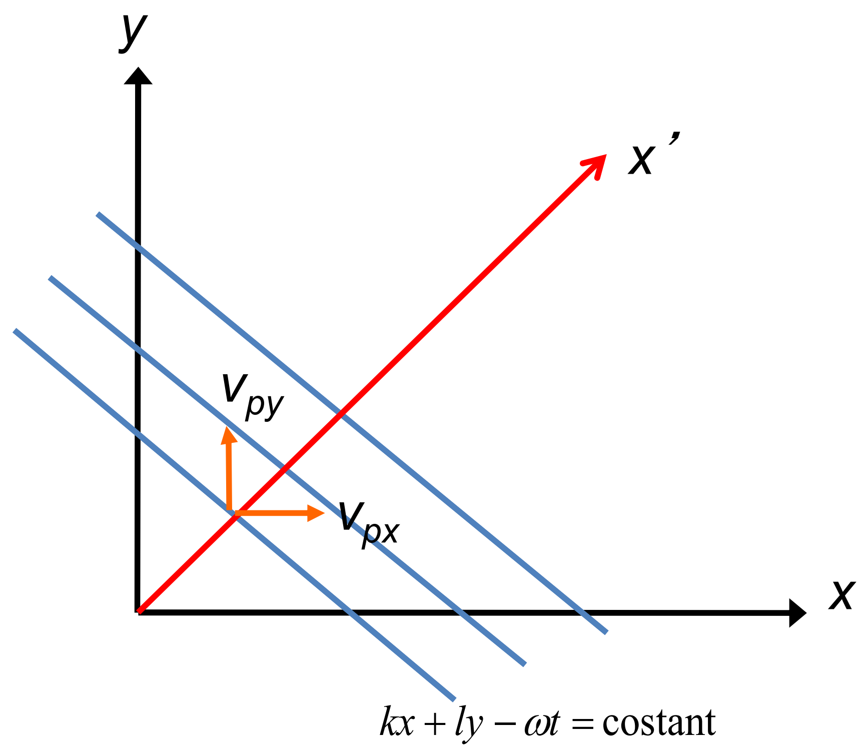
They differ from the x and y component of v_p

$$\tilde{v}_{px} = v_p \frac{k}{\sqrt{k^2 + l^2}} = \frac{\omega}{\sqrt{k^2 + l^2}} \frac{k}{\sqrt{k^2 + l^2}}$$

$$= \frac{\omega k}{k^2 + l^2} = \frac{\omega}{k + \frac{l^2}{k}} < \frac{\omega}{k} = v_{px}$$

$$\tilde{v}_{py} = v_p \frac{l}{\sqrt{k^2 + l^2}} = \frac{\omega}{\sqrt{k^2 + l^2}} \frac{l}{\sqrt{k^2 + l^2}}$$

$$= \frac{\omega l}{k^2 + l^2} = \frac{\omega}{l + \frac{k^2}{l}} < \frac{\omega}{l} = v_{py}$$



The rule for the vector calculation is not applicable to determine the x and y components of phase speed.

Short-wave approximation

$k_oH >> 1$, $H >> 1/k_o \sim L$: Deep-water waves

In this approximation, $\tanh(k_oH) \approx 1$, $\cosh[k_o(z+H)] \approx \frac{1}{2}e^{k_o(z+H)}$

$$p' = A_o e^{i(kx+ly-\omega t)} \cosh[(z+H)\sqrt{k^2+l^2}] \approx \frac{A_o}{2} e^{i(kx+ly-\omega t)+k_o(z+H)};$$

$$\zeta = \frac{A_o}{g\rho_o} e^{i(kx+ly-\omega t)} \cosh[H\sqrt{k^2+l^2}] \approx \frac{A_o}{2g\rho_o} e^{i(kx+ly-\omega t)+k_o(z+H)}$$

$$w = -\frac{i\omega A_o}{g\rho_o} e^{i(kx+ly-\omega t)} \cosh[H\sqrt{k^2+l^2}] \approx -\frac{i\omega A_o}{2g\rho_o} e^{i(kx+ly-\omega t)+k_o H}$$

$$\omega^2 = g\sqrt{k^2+l^2}\tanh(H\sqrt{k^2+l^2}) \approx g\sqrt{k^2+l^2} = gk_o$$

The phase speed equals

$$v_p = \frac{\omega}{k_o} = \pm\frac{\sqrt{gk_o}}{k_o} = \pm\sqrt{\frac{g}{k_o}}$$

Suppose that $l = 0$ (or we could choose a coordinate ststem in which $l = 0$)

$$p' = \mathrm{A}e^{i(kx-\omega t)+kz}$$

$$v = 0$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} = -\frac{ikA}{\rho_o}e^{i(kx-\omega t)+kz}$$

$$\Rightarrow u = u_o(x,z)e^{-i\omega t} \Rightarrow -i\omega u_o(x,z)e^{-i\omega t} = -\frac{ikA}{\rho_o}e^{i(kx-\omega t)+kz}$$

$$u_o(x,z) = \frac{kA}{\omega\rho_o}e^{ikx+kz} \Rightarrow u = u_o(x,z)e^{-i\omega t} = \boxed{\frac{kA}{\omega\rho_o}e^{i(kx-\omega t)+kz}}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} = -\frac{kA}{\rho_o}e^{i(kx-\omega t)+kz} \Rightarrow w = \boxed{\frac{kA}{i\omega\rho_o}e^{i(kx-\omega t)+kz}}$$

$$\zeta = \frac{A}{g\rho_o}e^{i(kx-\omega t)}$$

The real parts of the solution are given as

$$\begin{cases} p' = A\,\mathbf{cos}(kx - \omega t)e^{kz} \\ \zeta = \dfrac{A}{g\rho_o}\mathbf{cos}(kx - \omega t) \\ u = \dfrac{A}{\rho_o}(\dfrac{k}{\omega})\,\mathbf{cos}(kx - \omega t)e^{kz} \\ w = \dfrac{A}{\rho_o}(\dfrac{k}{\omega})\,\mathbf{sin}(kx - \omega t)e^{kz} \end{cases}$$

The short-surface waves are trapped near the surface over a e-folding vertical scale of 1/k. Therefore, the propagation of waves is not affected by the bottom

Particle trajectories

$$x - x_o = -\frac{A}{\rho_o}(\frac{k}{\omega^2})\,\mathbf{sin}(kx - \omega t)e^{kz}; \;\; z - z_o = \frac{A}{\rho_o}(\frac{k}{\omega^2})\,\mathbf{cos}(kx - \omega t)e^{kz}$$

$$(x - x_o)^2 + (z - z_o)^2 = [\frac{A}{\rho_o}(\frac{k}{\omega^2})e^{kz}]^2$$

Particle trajectory in a deep-water wave is a circle and its radius decreases with depth.

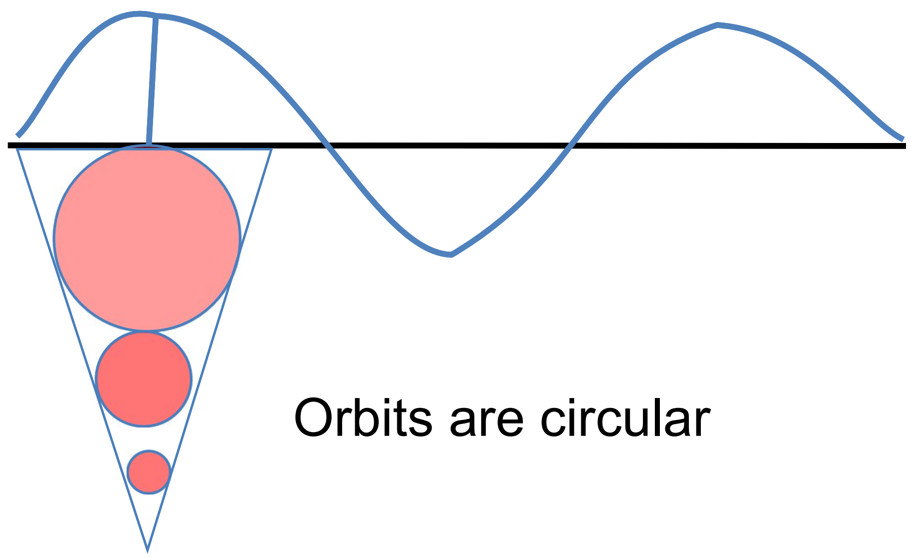
The size of circle ~ A; k/ω; z;

The phase of waves propagates, but particles don't;

Average over a wave period, the velocity equals zero;

The phase speed is a function of the wave number but does not depends on the water depth;

The waves are trapped near the surface. In this deep-water, the hydrostatic balance is not valid.



Orbits are circular

Long-wave approximation

$k_oH << 1$, $H << 1/k_o \sim L$: Shallow-water waves

In this approximation, $\tanh(k_oH) \approx k_oH - 1/3(k_oH)^3 \ldots \approx k_oH$

$$\omega^2 = g\sqrt{k^2+l^2}\,\mathbf{tanh}(H\sqrt{k^2+l^2}) \approx gk_o^2H \Rightarrow v_p = \frac{\omega}{k} = \pm\sqrt{gH}$$

$$p' = \mathrm{A}e^{i(kx+ly-\omega t)}[1+\frac{k_o}{2}(z+H)^2+\ldots] \approx Ae^{i(kx+ly-\omega t)};$$

$$\zeta = \frac{\mathrm{A}}{g\rho_o}e^{i(kx+ly-\omega t)}[1+\frac{k_o}{2}(z+H)^2+\ldots] \approx \frac{\mathrm{A}}{g\rho_o}e^{i(kx+ly-\omega t)}$$

$$u = \mathrm{A}(\frac{k}{\omega})e^{i(kx+ly-\omega t)}[1+\frac{k_o}{2}(z+H)^2+\ldots] \approx \mathrm{A}(\frac{k}{\omega})e^{i(kx+ly-\omega t)}$$

$$w = \mathrm{A}(\frac{k}{i\omega})e^{i(kx+ly-\omega t)}[k_o(z+H)+\ldots] \approx \mathrm{A}(\frac{k}{i\omega})e^{i(kx+ly-\omega t)}\mathrm{k_o}(z+H)$$

Particle trajectories

$$\frac{(x-x_o)^2}{(\frac{Ak}{\omega^2})^2}+\frac{(z-z_o)^2}{[\frac{Ak}{\omega^2}k(z+H)]^2}=1$$

The particle's trajectory is an ellipse

The pressure and horizontal velocity are independent of depth, while the vertical velocity is a linear function of depth;

The phase speed is determined by total water depth, but independent of the wave number;

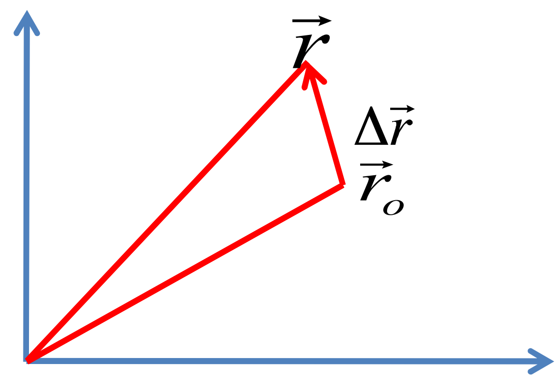
In the shallow water limit, the hydrostatic balance is valid

Stoke's drift velocity

The conclusion for the particle trajectories of the deep-water waves is only valid for a linear model. The linear theory describes the wave motion of "sinusoidal" function in the propagation direction. In the field of waves, the average velocity of particle equals to zero. However, it is not true for finite-amplitude gravity waves in a nonlinear system. In such a system, the particle orbits are not closed and there is a slow mean drift of the fluid elements in the direction of wave propagation: **stokes' drift**!

Let $u_l(r_o,t) \rightarrow$ the velocity of a fluid element whose position at time t is given by $x = r_o$.
After a time interval of Δt, the fluid element arrives at a new position at $r = r_o + \Delta r$.
According to the relation between Eulerian and Lagrangian velocities, we have

$$u_l(r,t) = u_E(r_o,t) + \Delta r \cdot \nabla u(r_o,t)$$



Suppose in an oscillated motion, the fluid element still keep its position near the initial location

$$\Delta r = \int_t^{t+\Delta t} u(r_o,t)\,dt$$

$$u_l(r,t) = u_E(r_o,t) + \left(\int_t^{t+\Delta t} u(r_o,t)\,dt\right)\cdot \nabla u(r_o,t)$$

Averaging over a wave period, we have

$$\overline{u_l(r,t)} = \overline{u_E(r_o,t)} + \overline{\left(\int_t^{t+\Delta t} u(r_o,t)\,dt\right)\cdot \nabla u(r_o,t)} = \overline{U}_E + \boxed{\overline{U}_s}$$

Stokes’ drift velocity

In the zero-order approximation,

$$u_E = u_l$$

In the first-order approximation,

$$u_E \neq u_l, \quad \text{the stokes' velocity } u_s \neq 0$$

The Surface Wave Model (SWAN or FVCOM-SWAVE)

(references: Booij et al. 1999, SWAN Team, 2006, Qi et al. 20008)

The evolution of wave spectra is determined by the wave action density spectrum balance equation expressed as

$$\frac{\partial N}{\partial t} + \nabla \cdot [(\vec{C}_g + \vec{V}) N] + \frac{\partial C_\sigma N}{\partial \sigma} + \frac{\partial C_\theta N}{\partial \theta} = \frac{S_{tot}}{\sigma}$$

where N is the wave action density spectrum; t is the time; σ is the relative frequency; θ is the wave direction.

Two spaces: the spectral space (σ, θ) and in geographic space (x, y)

$$S_{tot} = S_{in} + S_{nl3} + S_{nl4} + S_{ds,w} + S_{ds,b} + S_{ds,br}$$

where S_{in} is the function for the wind-induced wave growth; S_{nl3} is the nonlinear transfer of wave energy due to three-wave interactions; S_{nl4} is the nonlinear transfer of wave energy due to four-wave interactions; $S_{ds,w}$ is the wave decay due to white capping; $S_{ds,b}$ is the wave decay due to bottom friction; and $S_{ds,br}$ is the wave decay due to depth-induced wave breaking.

Discrete Algorithms:

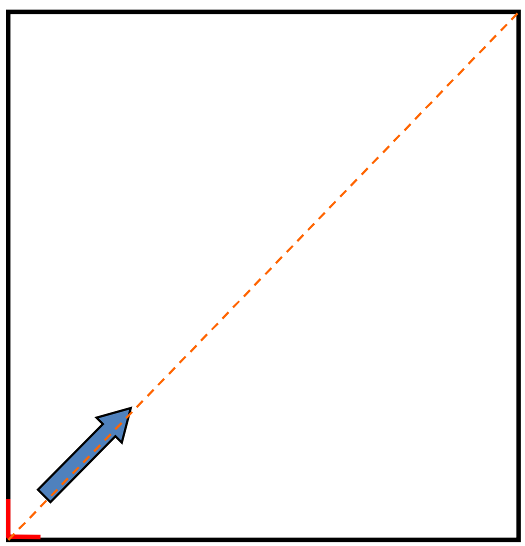
SWAN (Simulating Waves Nearshore): The structured grid (rectangular or curvilinear) and is solved by implicit schemes in both spectral and geographic spaces (SWAN Team, 2006);

FVCOM-SWAVE: The unstructured grid (triangular) and is solved by the Flux-Corrected Transport (FCT) algorithm in frequency space; the implicit Crank-Nicolson method in directional space and options of explicit or implicit second-order upwind finite-volume schemes in geographic space (Qi et al., 2008). FVCOM-SWAVE is the unstructured grid of SWAN at the second-order accuracy.

UnSWAN: A new unstructured grid version of SWAN developed by the SWAN Team. At present, it is solved using the first-order accurate discrete scheme.

FE-WAVE: A unstructured grid version of SWAN solved using the finite-element method (Hsu et a., 2005). This is not an open-source code.

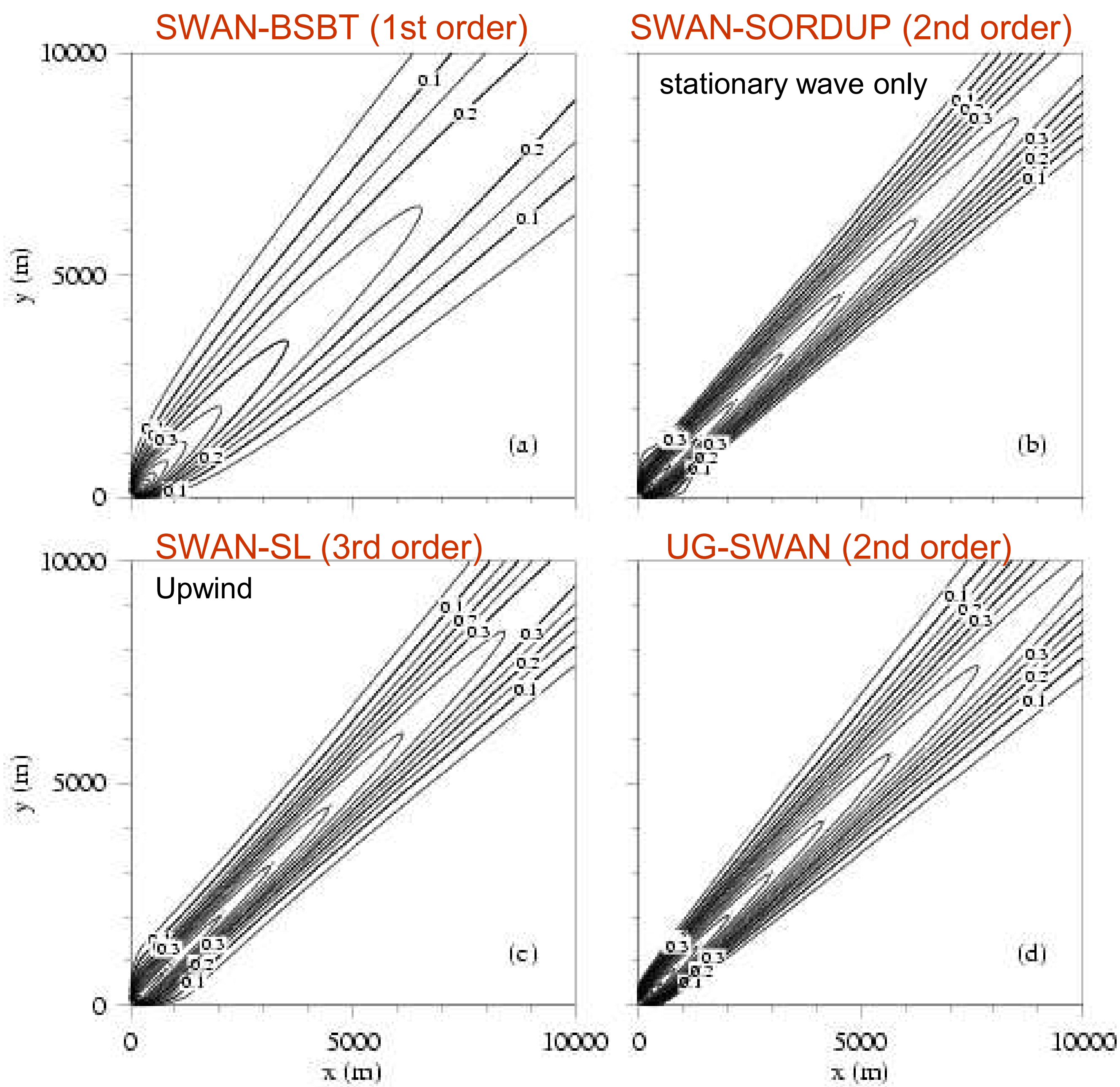
Test 1: Dispersion experiments



Hs (Significant wave height) = 1 m

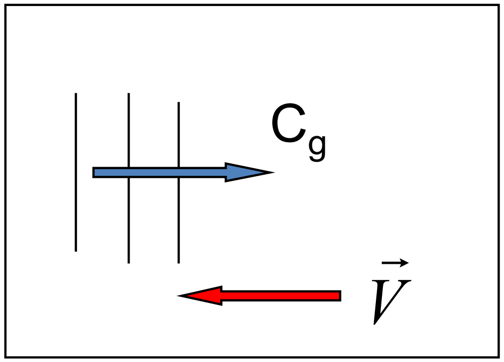
σ (frequency) = 0.1 Hz

Examine the numerical dispersion for the wave propagation

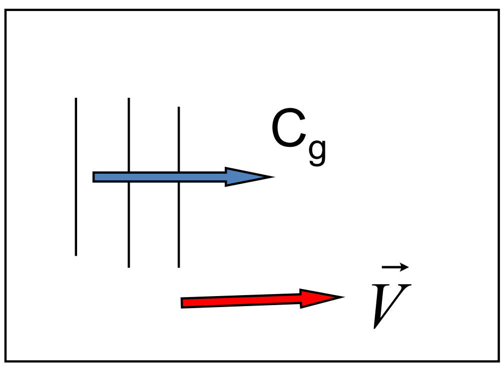


Test 2-a: Wave and Current Interaction

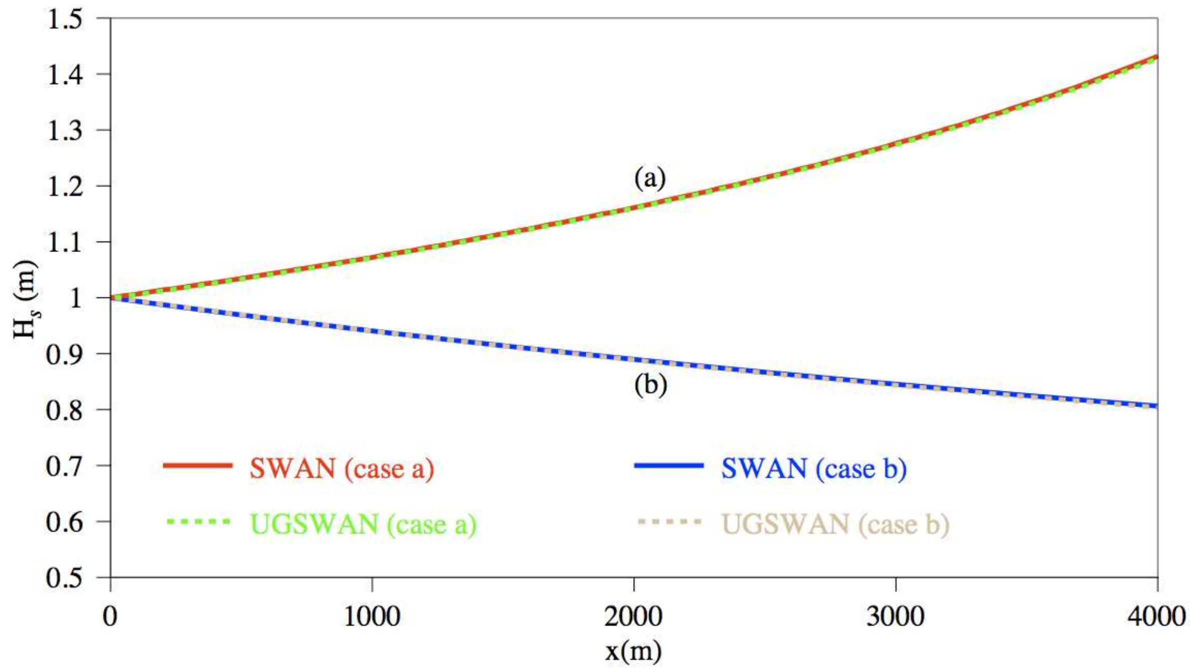
Case a:



Case b:

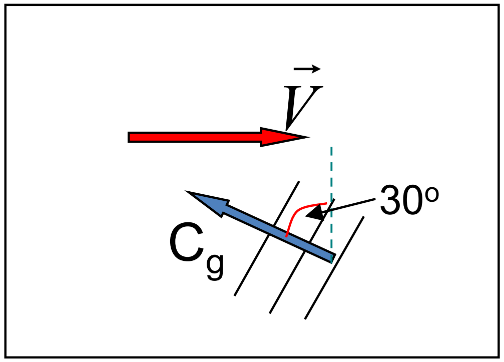


Results for significant wave height

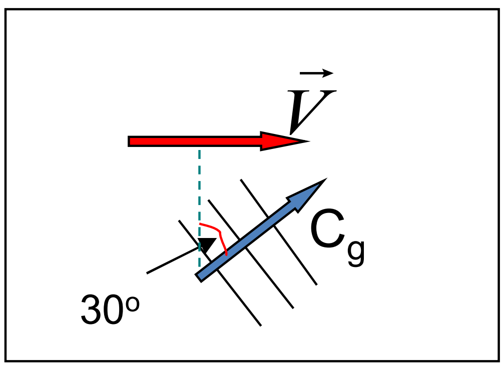


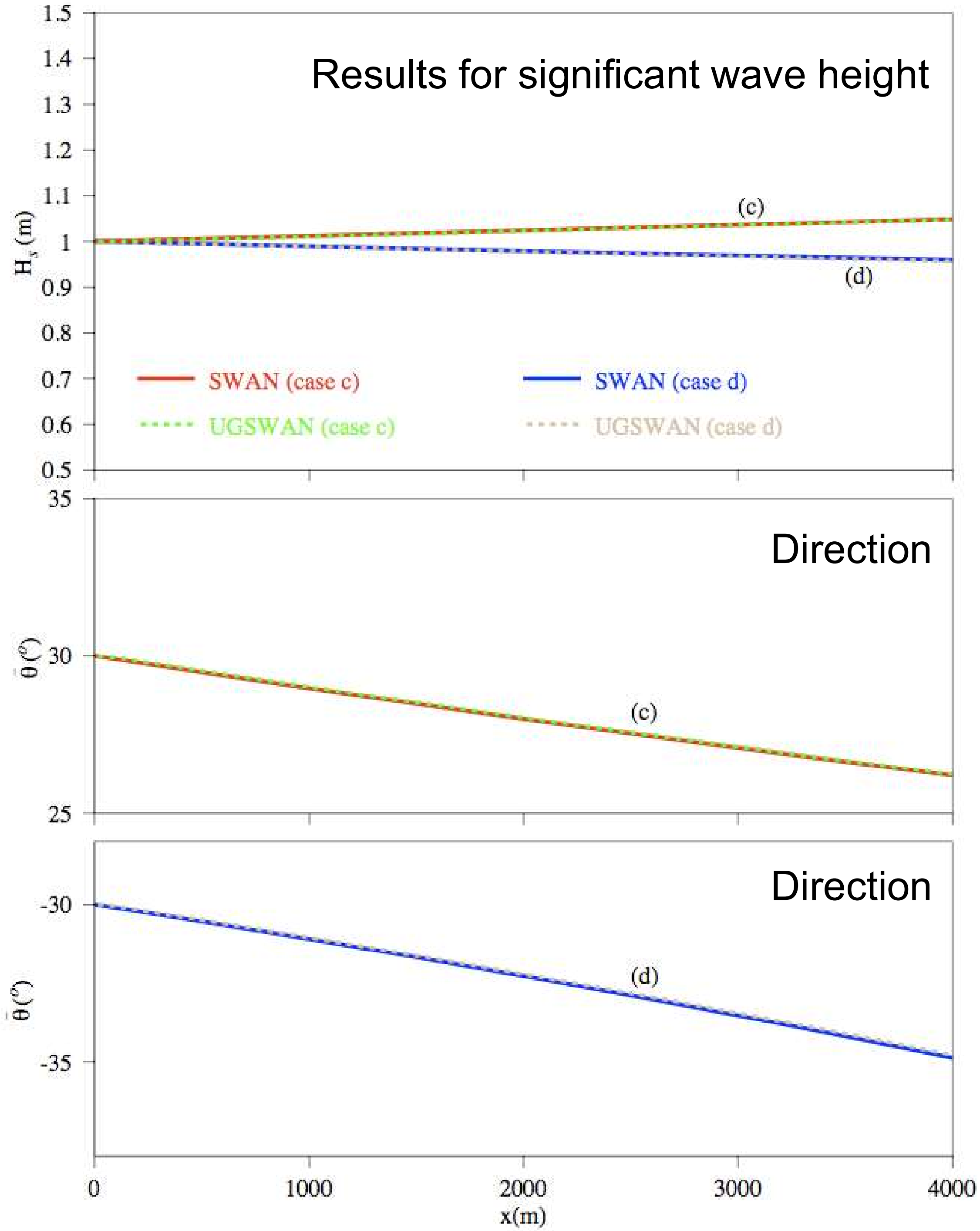
Test 2-b: Wave and Current Interaction

Case c:



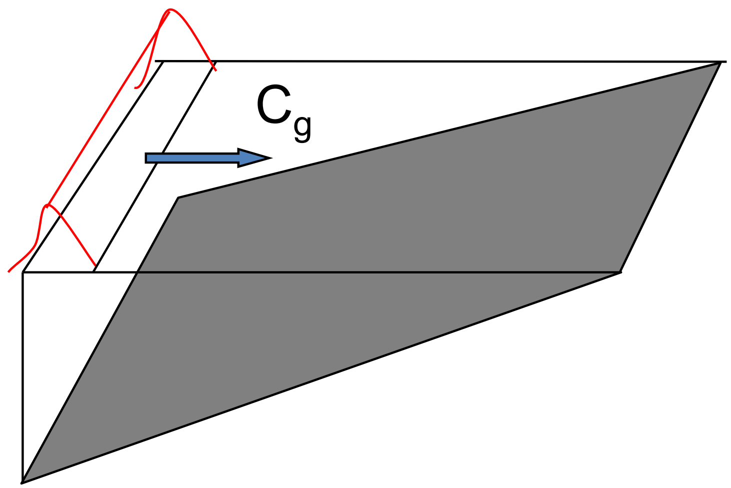
Case d:



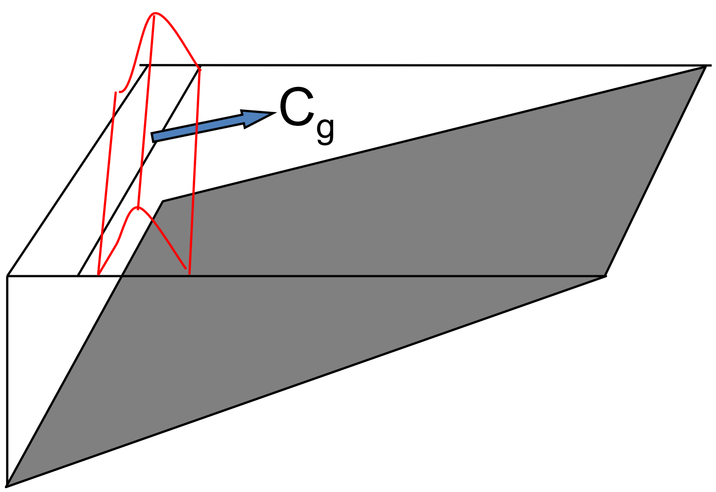


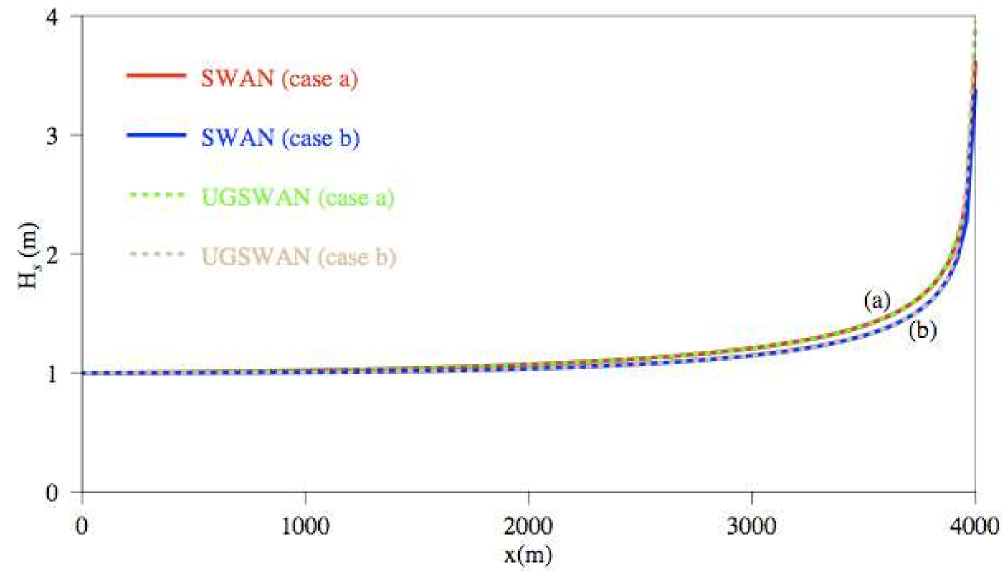
Test 3: Wave propagating onto the shelf

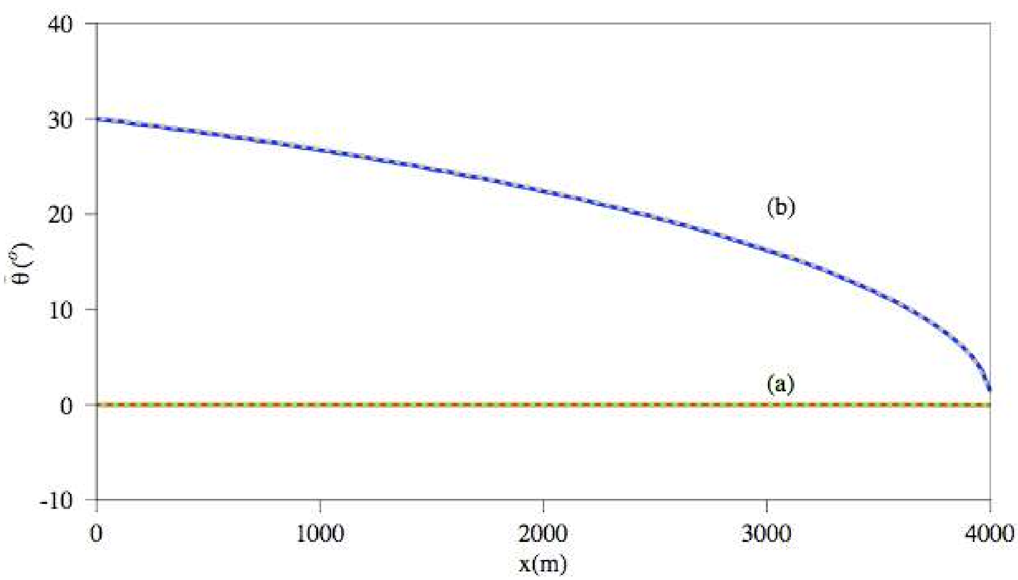
Case a:



Case b:

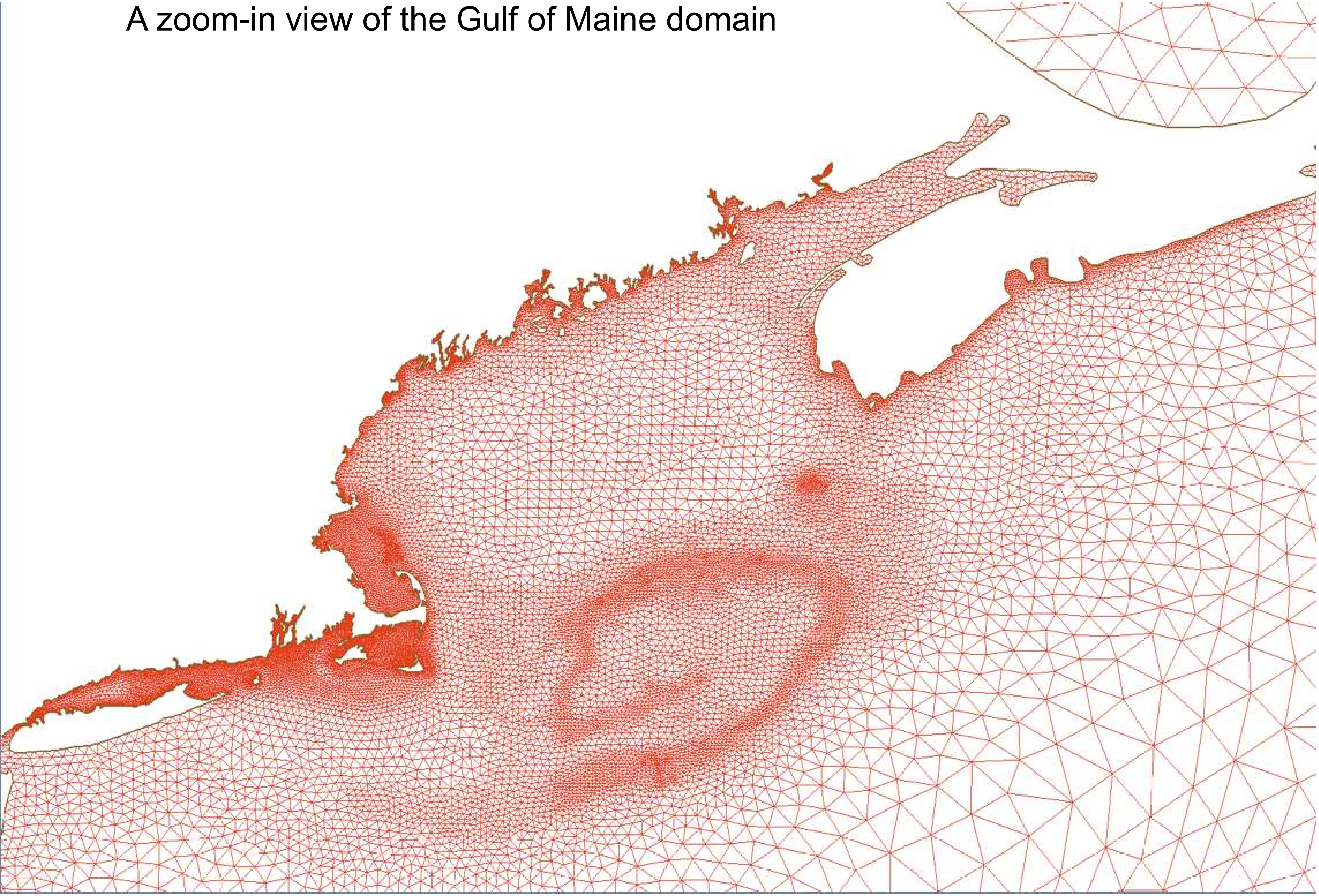






A Test Case for the Gulf of Maine

A zoom-in view of the Gulf of Maine domain



Forcing:

GOM-WRF hindcast wind fields (with a resolution of 9 km) [one component of the Northeast Coastal Ocean Forecast System (NECOFS] plus 32-km Atlantic Oceanic WRF produced wind fields;

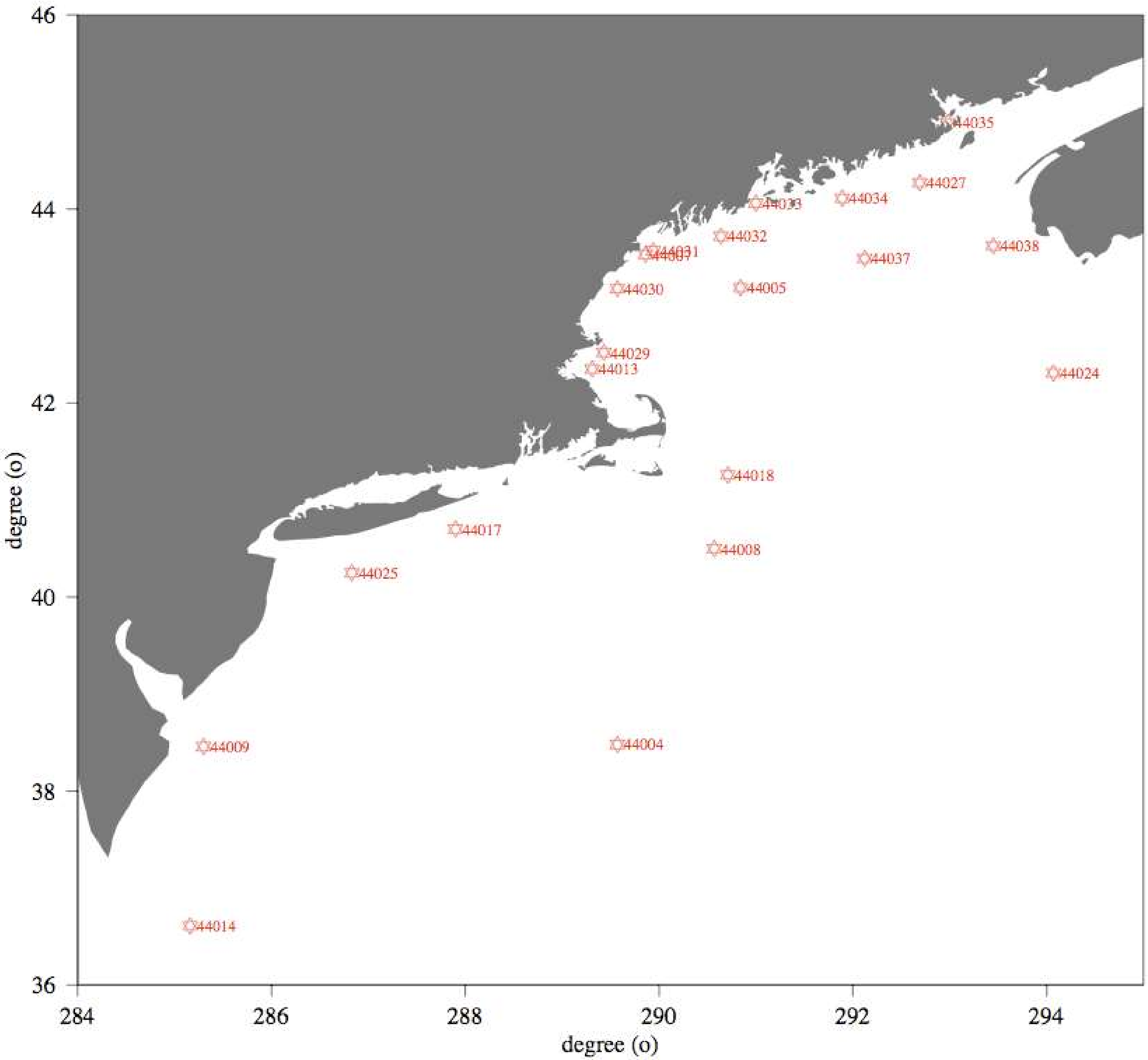
Wave growth function by wind: 1) Komen et al. (1984), 2) Janssen (1989, 1991) and 3) Yan (1987);

Whitecapping function: Komen default formula;

Period: January 1 2007 to January 30, 2007

Time step: 1.5 minutes

32 CPU: one computational days for 10 real days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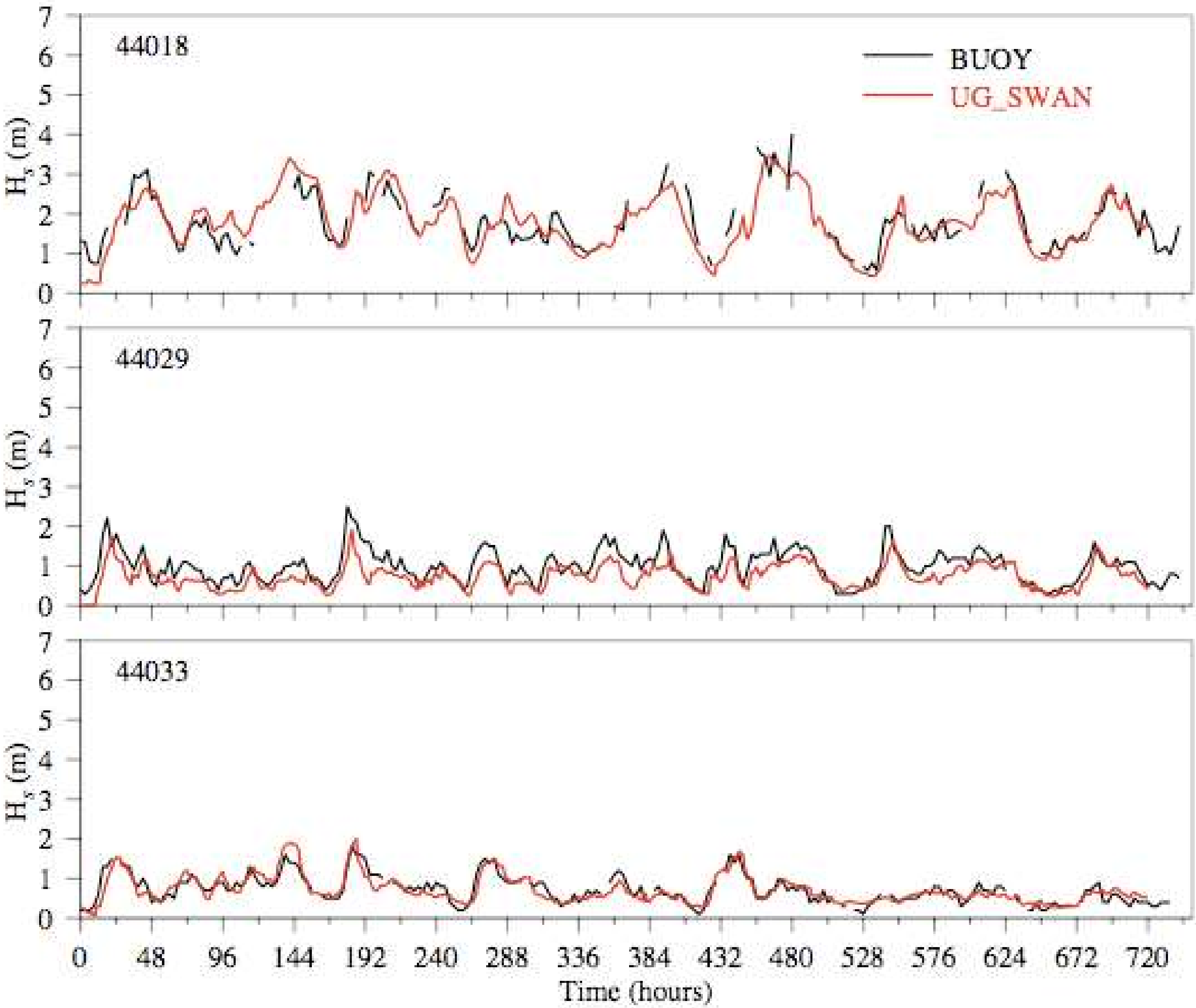
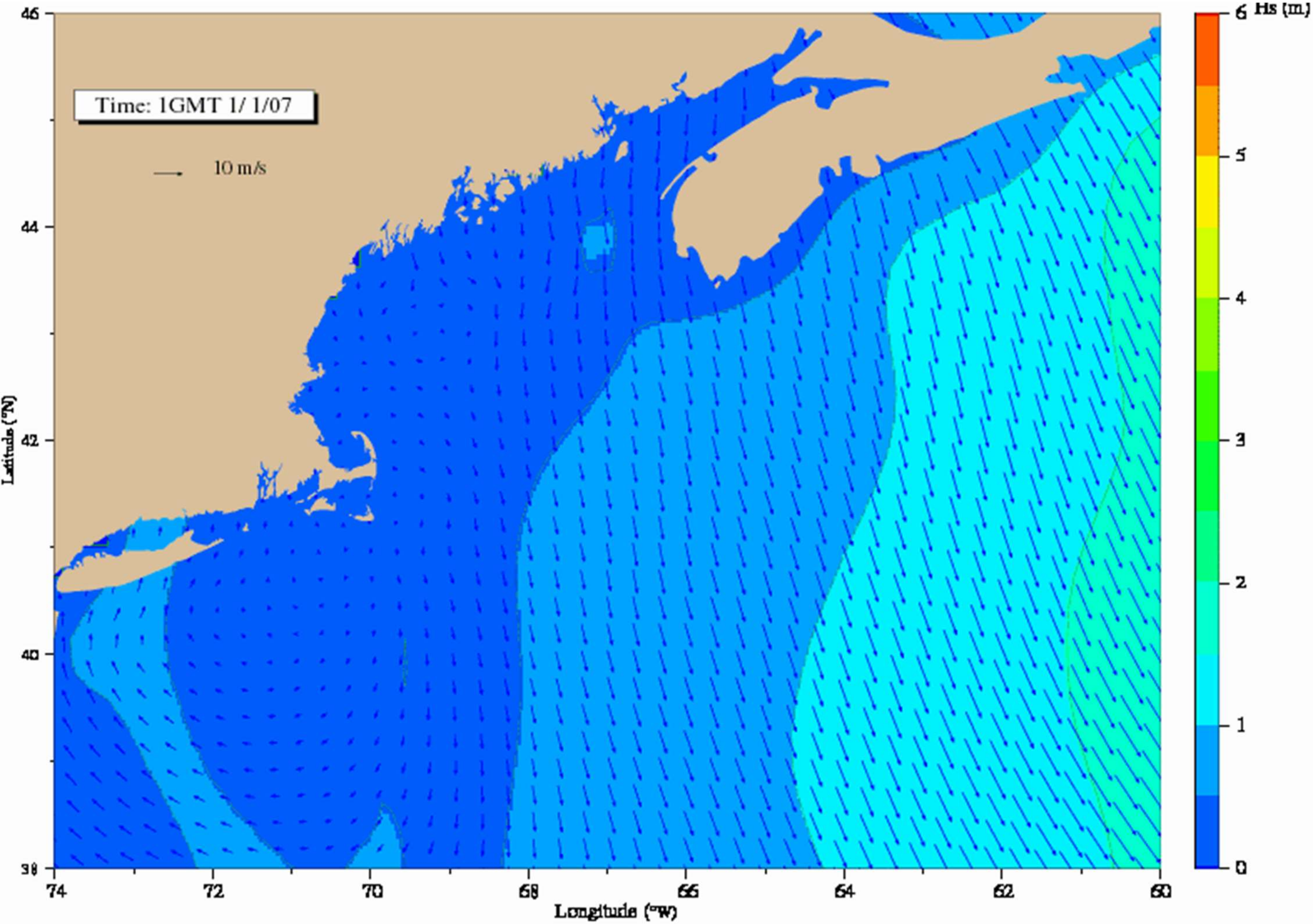
Komen growth function

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Latitude (°N)
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Longitude (°W)

Coupling of Hydrodynamics and Wave Models

The coupling of hydrodynamics (motions with a period longer than the surface waves) and waves is by the radiation stresses:

$$\frac{du}{dt} - fv = RHS(u) - \frac{\partial S_{xx}}{\partial x} - \frac{\partial S_{xy}}{\partial y} - \frac{\partial S_{xz}}{\partial z};$$

$$\frac{dv}{dt} + fu = RHS(v) - \frac{\partial S_{yx}}{\partial x} - \frac{\partial S_{yy}}{\partial y} - \frac{\partial S_{yz}}{\partial z}$$

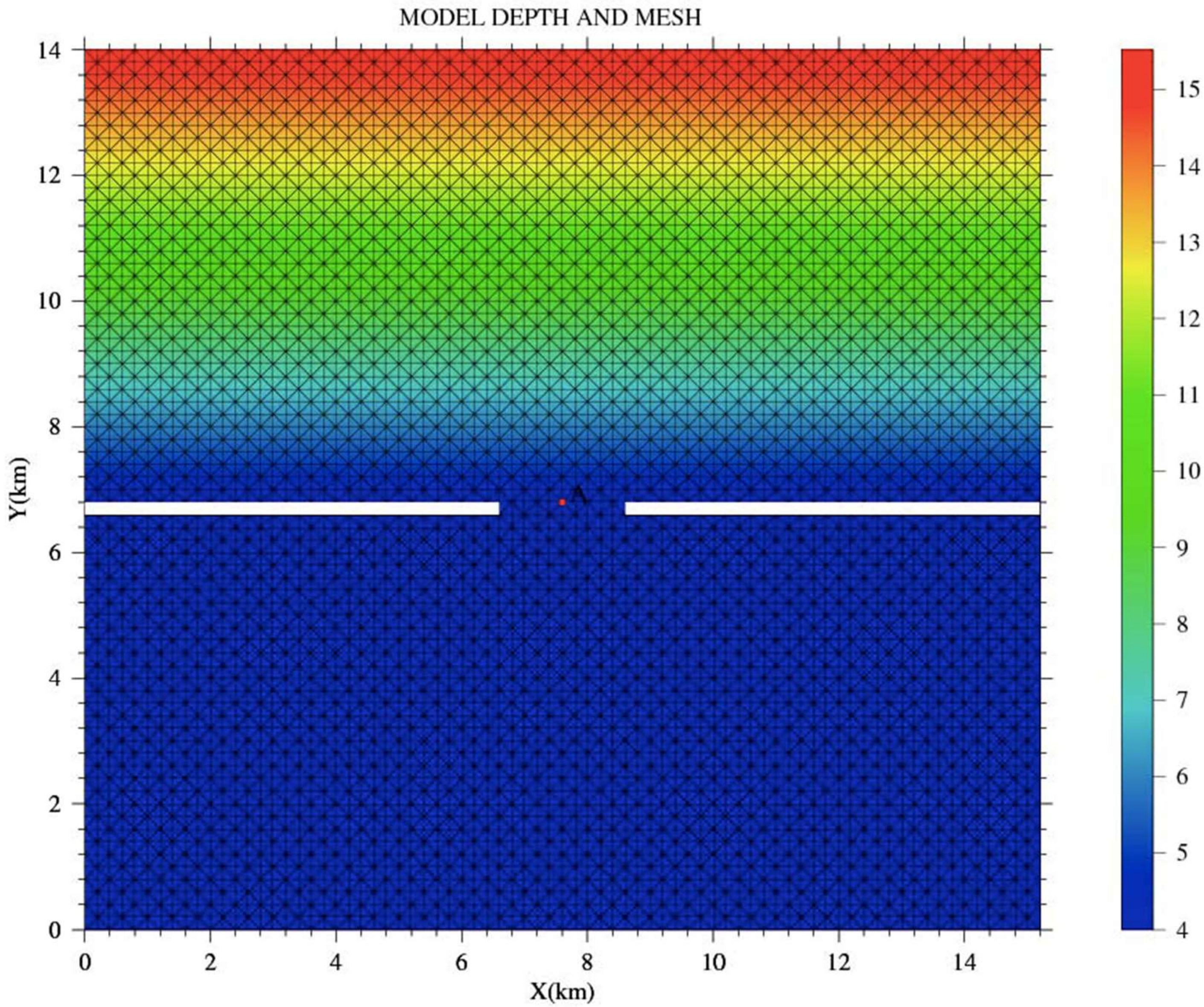
where

S_{xx}, S_{xy}, S_{xz} are the x, y and z component of the radiation stress in the u-momentum equation;

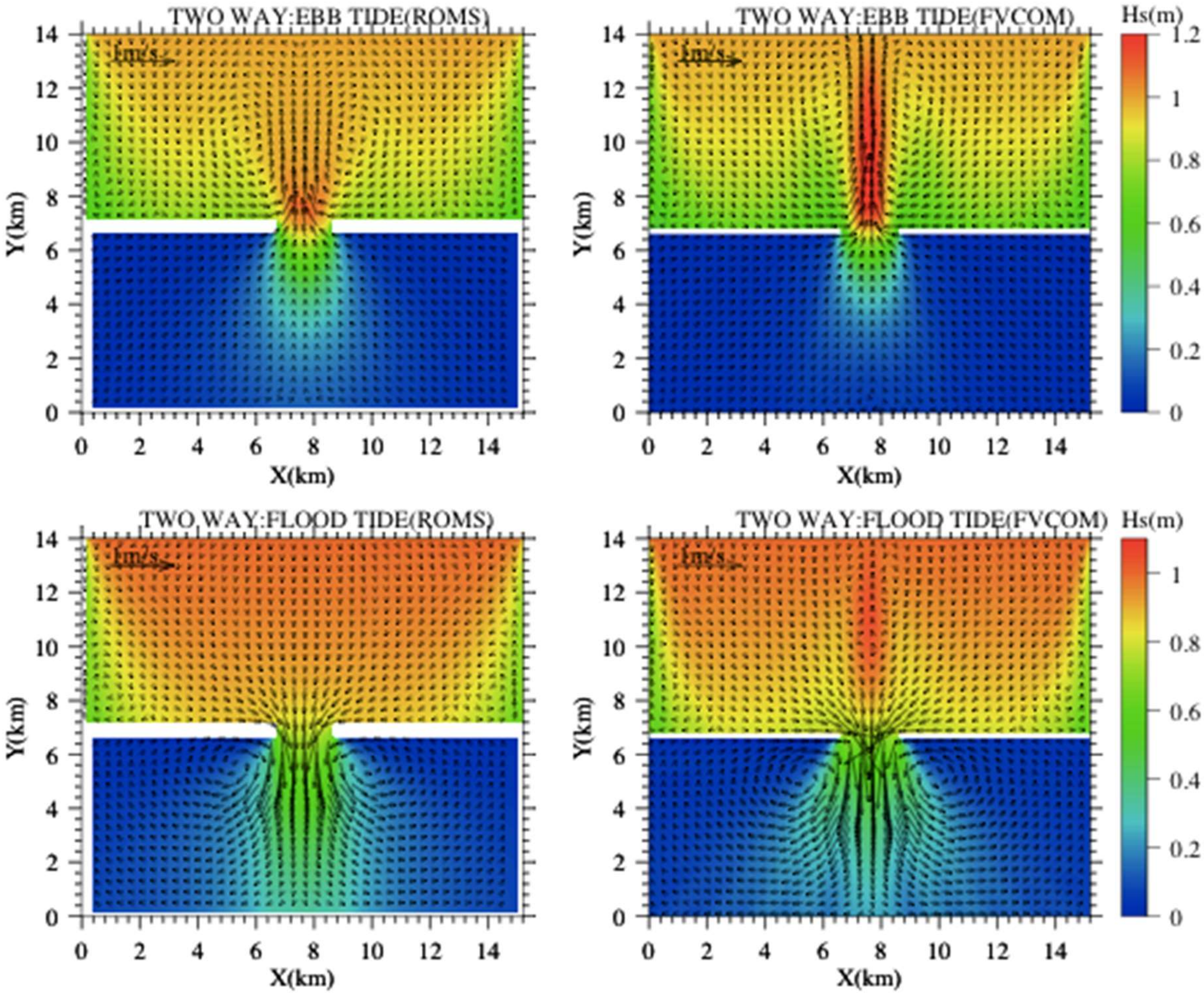
S_{yx}, S_{yy}, S_{yz} are the x, y and z component of the radiation stress in the v-momentum equation.

Reference: Warner et al. (2008).

The benchmark test for the coupled hydrodynamics-wave code



Experiments made by Wu


TWO WAY:EBB TIDE(ROMS)
TWO WAY:EBB TIDE(FVCOM)
TWO WAY:FLOOD TIDE(ROMS)
TWO WAY:FLOOD TIDE(FVCOM)
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X(km)
Y(km)

Bed thickness comparison