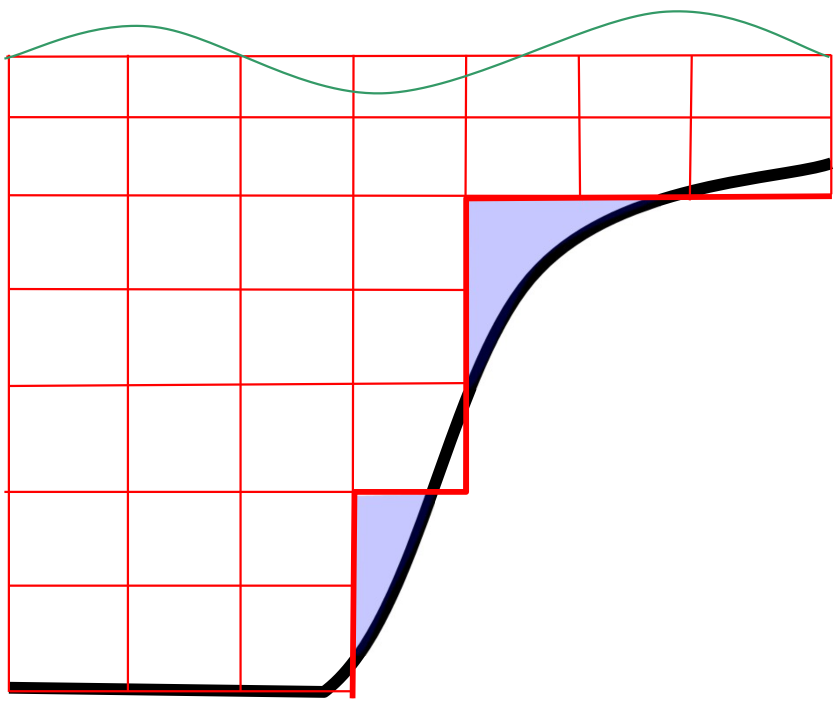
MAR513 Lecture 10: Pressure Errors in Terrain-Following Coordinates

The vertical coordinates:

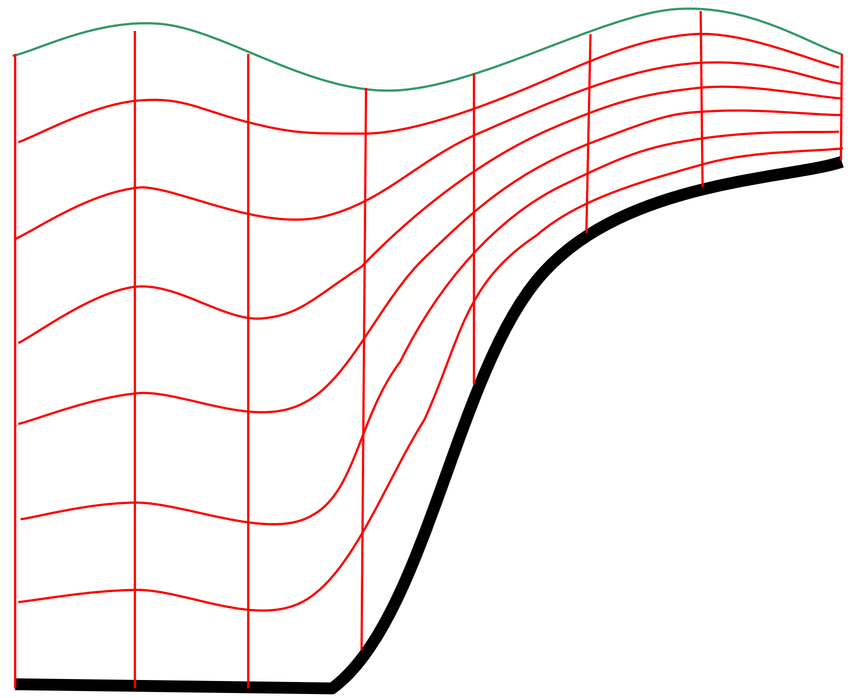
The z-coordinate



Advantage: Simple

Disadvantage: Poorly resolve topography

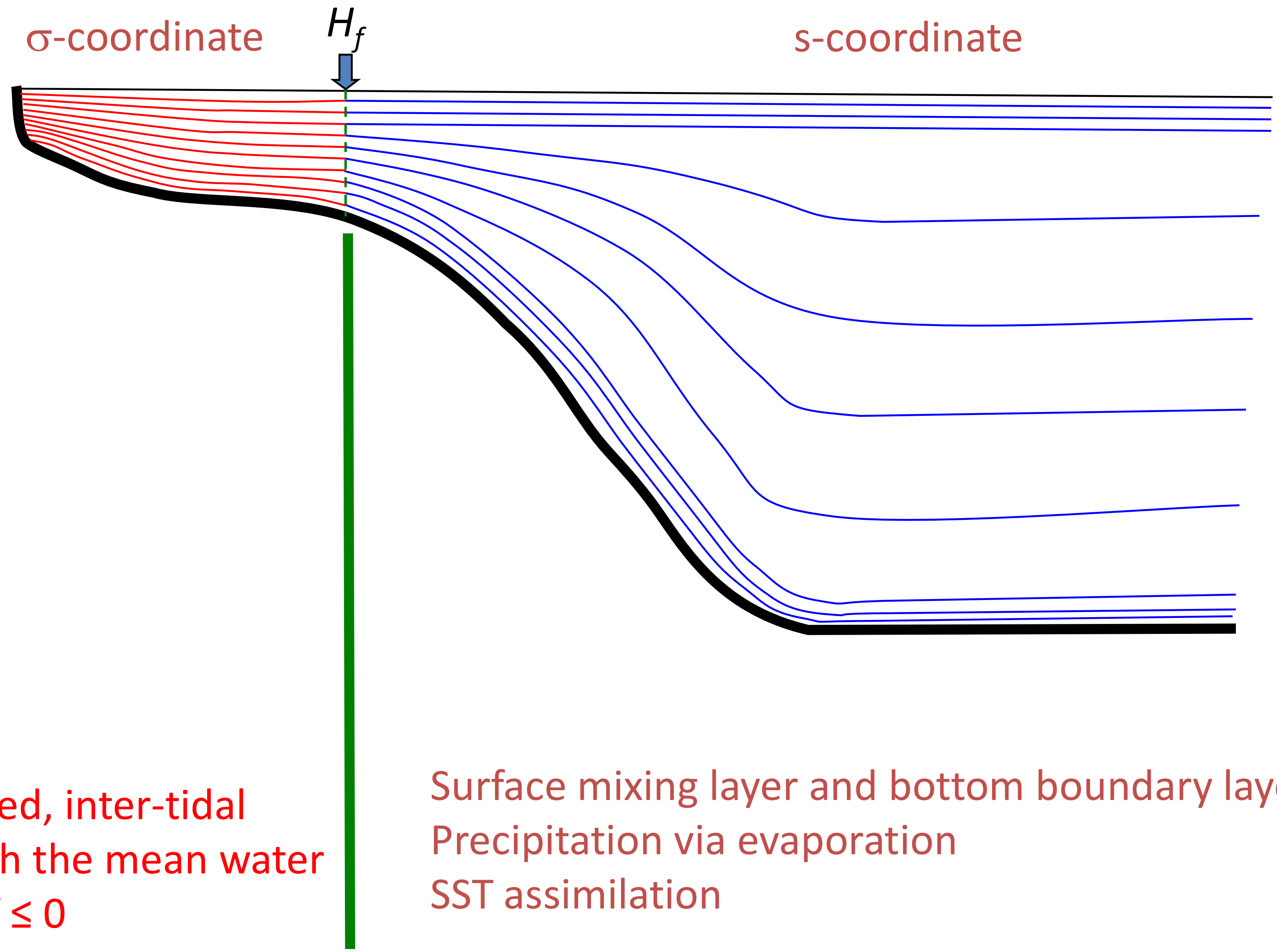
The Terrain-following coordinate



Advantage: Accurately resolve topography

Disadvantage: Pressure errors

The hybrid coordinate



Watershed, inter-tidal zone with the mean water depth of ≤ 0

Surface mixing layer and bottom boundary layer
Precipitation via evaporation
SST assimilation

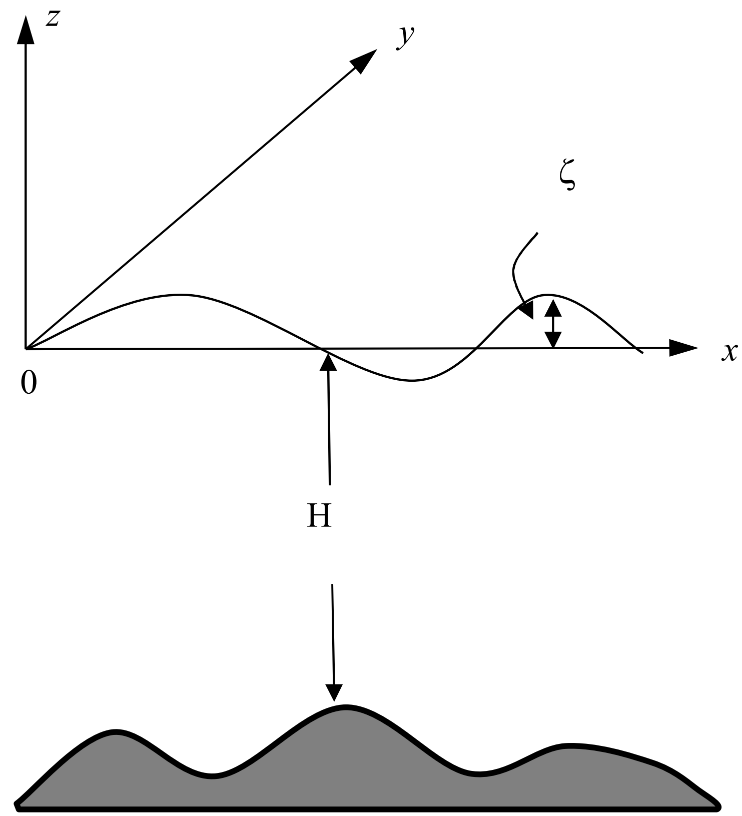
Pressure error in the σ-coordinate

An example based on the σ-coordinate:

$$\sigma = \frac{z-\zeta}{H+\zeta} = \frac{z-\zeta}{D}$$

After the transformation, the vertical domain will have a range of (0, -1)

$$-\frac{1}{\rho_o}\frac{\partial P}{\partial x} = -\frac{1}{\rho_o}\left[\frac{\partial P}{\partial x} + \frac{\partial P}{\partial \sigma}\frac{\partial \sigma}{\partial x}\right]$$



In the hydrostatic balance,

$$\frac{\partial P}{\partial z} = -\rho g$$

Integrating from z to ζ

$$P = P_a + \int_z^{\zeta} \rho g dz$$

In the σ-coordinate

$$z = \sigma D + \zeta \Rightarrow \partial z = D\partial\sigma; \quad \frac{\partial\sigma}{\partial x} = -\frac{1}{D}\left(\sigma\frac{\partial D}{\partial x} + \frac{\partial\zeta}{\partial x}\right)$$

$$\frac{\partial P}{\partial z} = -\rho g \Rightarrow \frac{\partial P}{D\partial\sigma} = -\rho g \Rightarrow \frac{\partial P}{\partial\sigma} = -\rho g D$$

$$P = P_a + \int_\sigma^0 \rho g D d\sigma \Rightarrow P_b = \int_\sigma^0 \rho g D d\sigma$$

Let us only consider the pressure in the water, we have

$$\frac{\partial P_b}{\partial x} = \frac{\partial P_b}{\partial x} + \frac{\partial P_b}{\partial\sigma}\frac{\partial\sigma}{\partial x} = \frac{\partial}{\partial x}\left(\int_\sigma^0 \rho g D d\sigma\right) + \frac{\partial}{\partial\sigma}\left(\int_\sigma^0 \rho g D d\sigma\right)\frac{\partial\sigma}{\partial x}$$

The final form of the pressure gradient can be written as

$$\frac{\partial P_b}{\partial x} = g\left(D\frac{\partial}{\partial x}\int_\sigma^0 \rho' d\sigma - \frac{\partial D}{\partial x}\int_\sigma^0 \sigma\frac{\partial\rho'}{\partial\sigma} d\sigma\right)$$

Where $\rho' = \rho - \rho_s(x, y, \sigma)$ and ρ_s is the reference density.

Note:

- After the σ-transformation, the pressure gradient term consists of two terms with opposite signs. This means that the pressure gradient in the σ-transformation coordinate system always equals to the difference of two terms.

- The two terms are in the same order of magnitude and the difference is a small value resulted from the difference of two big values.

This means that a small numerical error in these two big value terms could cause a big error in the numerical calculation of pressure gradient ! !

For example, A = B -C. Given B = 100 and C = 99, then A = 1

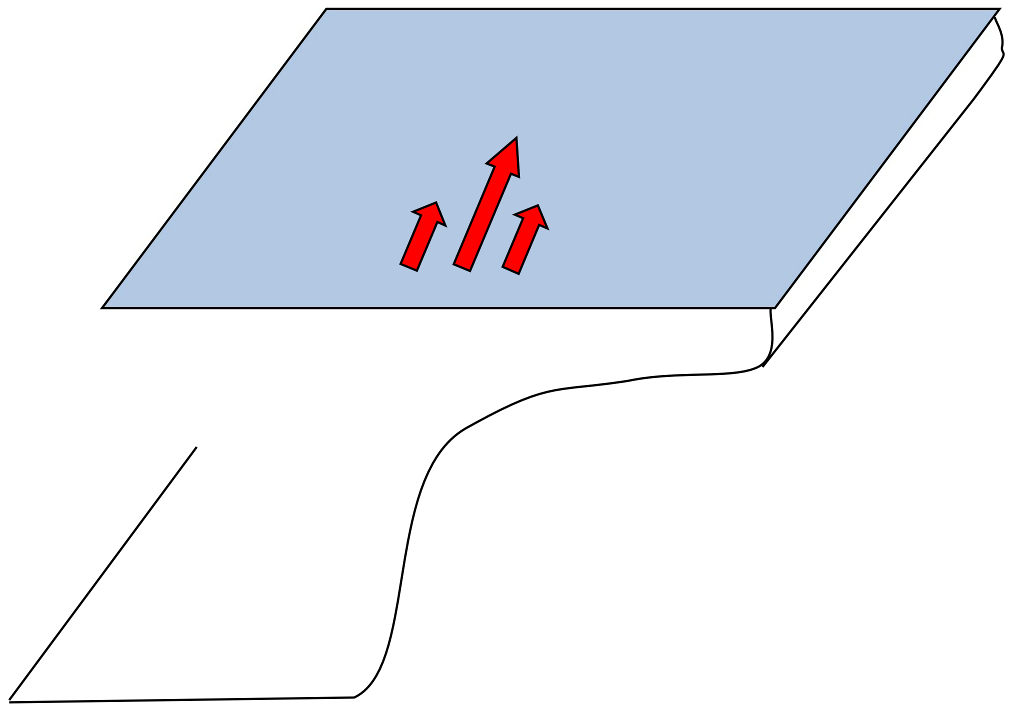
Now, let us add ~1% error in B and C,

B = 101, C= 98, then A = 101-98 = 3

In this case, the resulting error for A is

$$(3-1)/1 = 2 \Rightarrow 200\%$$

The error velocity likes the geostrophic motion: largest at the steepest topography. Since it is the same direction as the geostrophic current, it is difficult to distinguish it in the shelf break area where the shelf-break front is located.



QS: Could we develop a numerical scheme to avoid the numerical error in the pressure gradient calculation?

We might be able to design a scheme to reduce this error but it is difficult to have a method that can guarantee no error.

This issue has been addressed in many popular ocean models (like POM, FVCOM and ROMs) with different approaches.

For example:

1) Remove the mean density at each level;

2) Introduce a highly order approximation scheme;

3) Design the scheme to ensure no error in the vertical-linearly distribution of the density profile,

For example: Chen and Beardsley (1995)

Consider the *x*-*z* plane on which x is the cross-isobath direction and z is the vertical direction. Assume that the initial density is a linear function of z,

$$\rho\,(t=0) = \rho_0 - \left(\frac{\rho_b - \rho_0}{H_d}\right) z,$$

In this case,

$$\frac{\partial P_b}{\partial x} = \frac{\Delta P_b}{\Delta x} = 0$$

In the inviscid ocean, the σ-transformation will not produce the error motion over steep topography if only the perturbation density relative to the horizontal uniform reference density is used to calculate the pressure gradient. This is because

$$\rho' = \rho - \rho_s(x, y, \sigma) = 0$$

However, once diffusion is included or the system is forced externally, then

$$\rho' = \rho - \rho_s(x, y, \sigma) \neq 0$$

For the two-dimensional problem, the perturbation density is a function of *x*, *z* and *t,* and it can be rewritten as

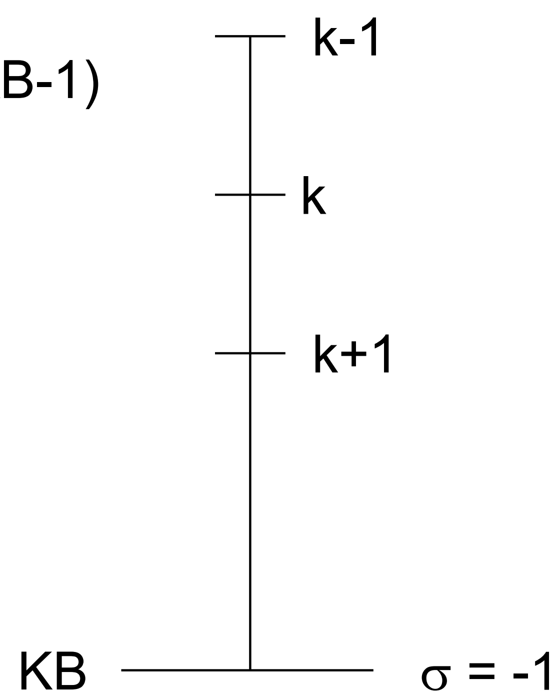
$$\rho' = \overline{\rho}'(z,t) + \rho''(x,z,t); \quad \overline{\rho''}(x,z,t) = 0$$

Let us use the numerical scheme as an example for this exercise

$$\frac{\Delta P_b}{\Delta x} = \frac{g}{\Delta x} \sum (D\Delta_x \rho' \Delta\sigma - \sigma \Delta_x D \Delta_\sigma \rho').$$

$$\begin{aligned}\left(\frac{\Delta P_b}{\Delta x}\right)_{i-(1/2),k+(1/2)} &= \left(\frac{\Delta P_b}{\Delta x}\right)_{i-(1/2),k-(1/2)} \\ &+ \frac{g}{\Delta x}\left\{\frac{1}{8}(\Delta\sigma_k + \Delta\sigma_{k-1})(D_i + D_{i-1})(\rho'_{i,k+(1/2)} \right. \\ &- \rho'_{i-1,k+(1/2)} + \rho'_{i,k-(1/2)} - \rho'_{i-1,k-(1/2)}) \\ &+ \frac{\sigma_k}{2}(D_i - D_{i-1})(\rho'_{i,k+(1/2)} + \rho'_{i-1,k+(1/2)} \\ &\left. - \rho'_{i,k-(1/2)} - \rho'_{i-1,k-(1/2)})\right\},\end{aligned}$$

k ={2, KB-1)



Let us assume that

$$\bar{\rho}' = \rho_0' + \hat{\beta}(-z)^n \quad \text{and} \quad z = H(x)\sigma,$$

then, we have

$$\left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k+(1/2)} = \left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k-(1/2)} + \frac{g(-1)^n}{\Delta x}\hat{\beta}\left\{\frac{1}{8}(\Delta\sigma_k + \Delta\sigma_{k-1})(D_i + D_{i-1}) \times (D_i^n - D_{i-1}^n)(\sigma_{k+(1/2)}^n + \sigma_{k-(1/2)}^n) + \frac{\sigma_k}{2}(D_i - D_{i-1})(D_i^n + D_{i-1}^n) \times (\sigma_{k+(1/2)}^n - \sigma_{k-(1/2)}^n)\right\}.$$

For n = 1

$$\left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k+(1/2)} = \left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k-(1/2)} + \frac{g\hat{\beta}}{16\Delta x}(D_i^2 - D_{i-1}^2)((\Delta\sigma_k)^2 - (\Delta\sigma_{k-1})^2).$$

If the uniform σ levels are used,

$\Delta\sigma = \Delta\sigma_k = \Delta\sigma_{k-1}$

$$\left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k+(1/2)} = \left(\frac{\Delta \bar{P}_b'}{\Delta x}\right)_{i-(1/2),k-(1/2)}$$

If the one could design a scheme to ensure no error in the first layer, then the model produce no error in all the layers.

For n =2

$$\left(\frac{\Delta \bar{P}'_b}{\Delta x}\right)_{i-(1/2),k+(1/2)} = \left(\frac{\Delta \bar{P}'_b}{\Delta x}\right)_{i-(1/2),k-(1/2)} + \frac{g\hat{\beta}\Delta\sigma\Delta D\bar{D}^2}{2\Delta x}\left\{\Delta\sigma^2 - \left(\frac{\Delta D}{\bar{D}}\right)^2 \sigma_k^2\right\}$$

where

$$\Delta D = (D_i - D_{i-1}) \text{ and } \bar{D} = (D_i + D_{i-1})/2$$

The error is proportional to the difference between two terms as follows

$$\left\{\Delta\sigma^2 - \left(\frac{\Delta D}{\bar{D}}\right)^2 \sigma_k^2\right\}$$

The first is independent of k and the second increases downward as k becomes large (k changes from 1 to KB)

In general,

$$\frac{\Delta D}{\bar{D}} > \frac{\Delta \sigma}{\sigma}$$

The maximum error is at the bottom. At $k = KB-1$,

$$\left(\frac{\Delta \bar{P}'}{\Delta x}\right)_{i-(1/2),KB-(1/2)} = -\frac{g\hat{\beta}\bar{D}^2}{2}\alpha(1-\Delta\sigma) \times \left\{\Delta\sigma^2 - \left(\frac{\alpha\Delta x}{\bar{D}}\right)^2 \frac{(2+\Delta\sigma)}{6}\right\}$$

$$\Delta D = -\alpha \Delta x$$

Then, the error velocity can be estimated by

$$V_e = \frac{1}{\rho_0 f}\left(\frac{\Delta \bar{P}'_b}{\Delta x}\right)_{KB-(1/2)}.$$

In a case with steep topography, for example, α ~0.03 (Georges Bank)

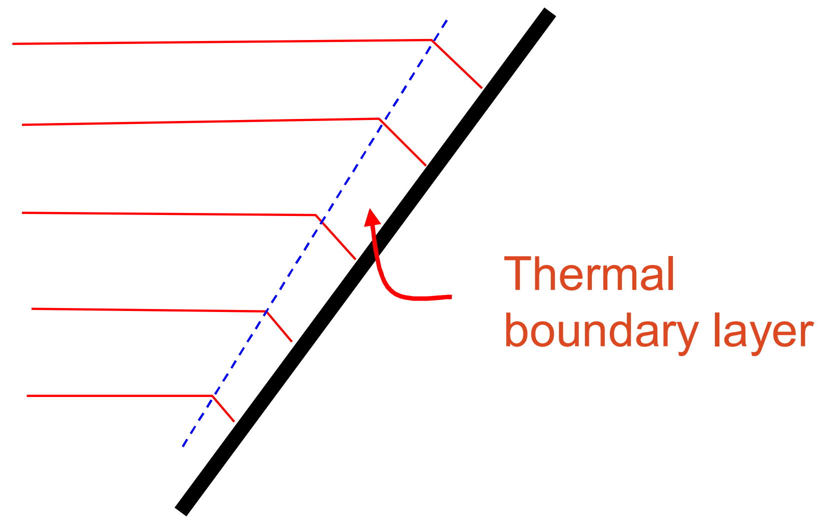
$$V_e \sim 0.2 \text{ cm/s}$$

This error is not very big. However, since the system is fully nonlinear, the big error velocity could be produced by the accumulation errors resulted from nonlinear interaction.

In the real ocean, diffusion over the slope tends to mix the water in the vertical, and the condition of no density flux into the slope bottom requires that the density contours encounters the slope at a right angle.

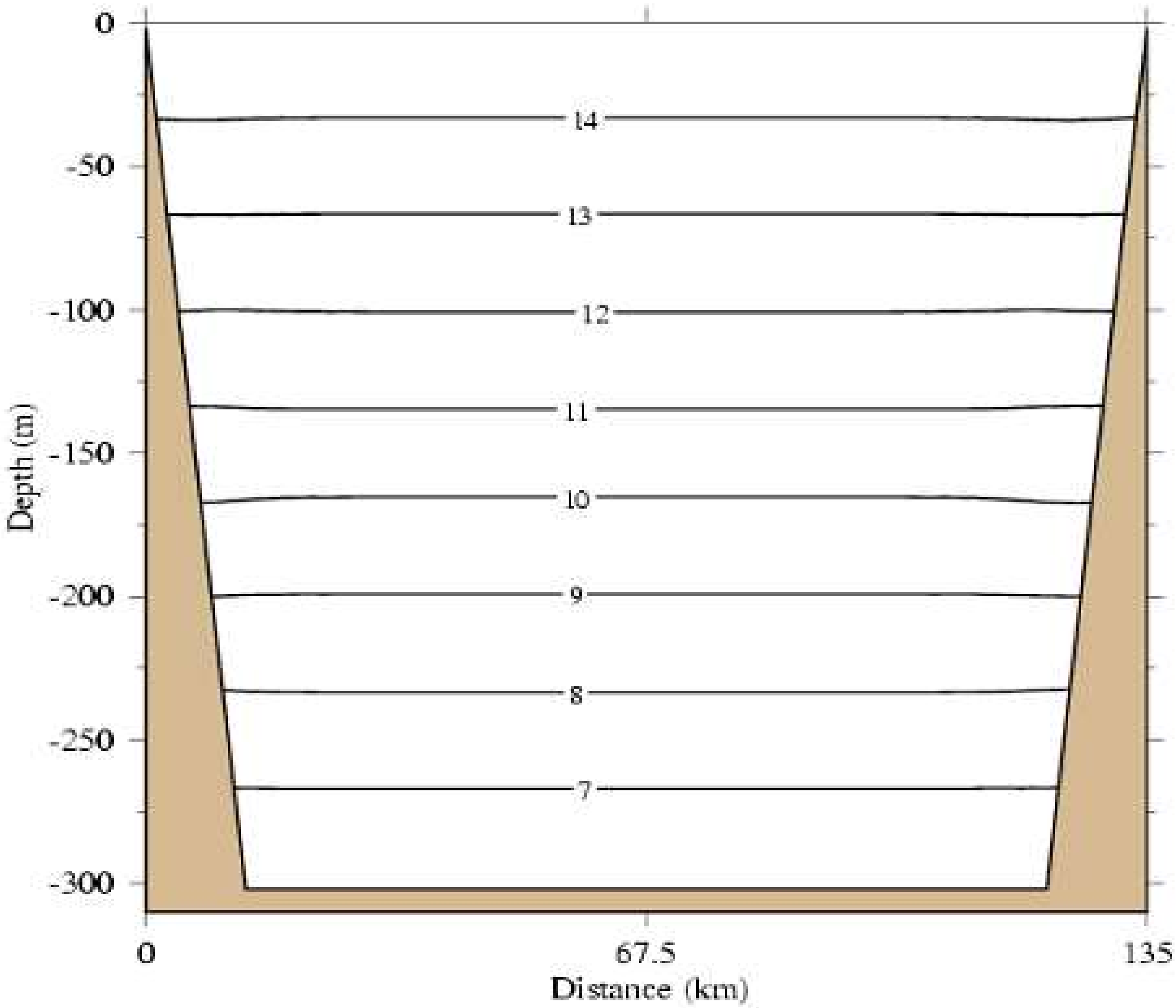
The thickness of the thermal boundary layer can be estimated by the expression derived by Wunsch (1970) as

$$\delta \sim (K_m K_H)^{1/4} N^{1/2}$$



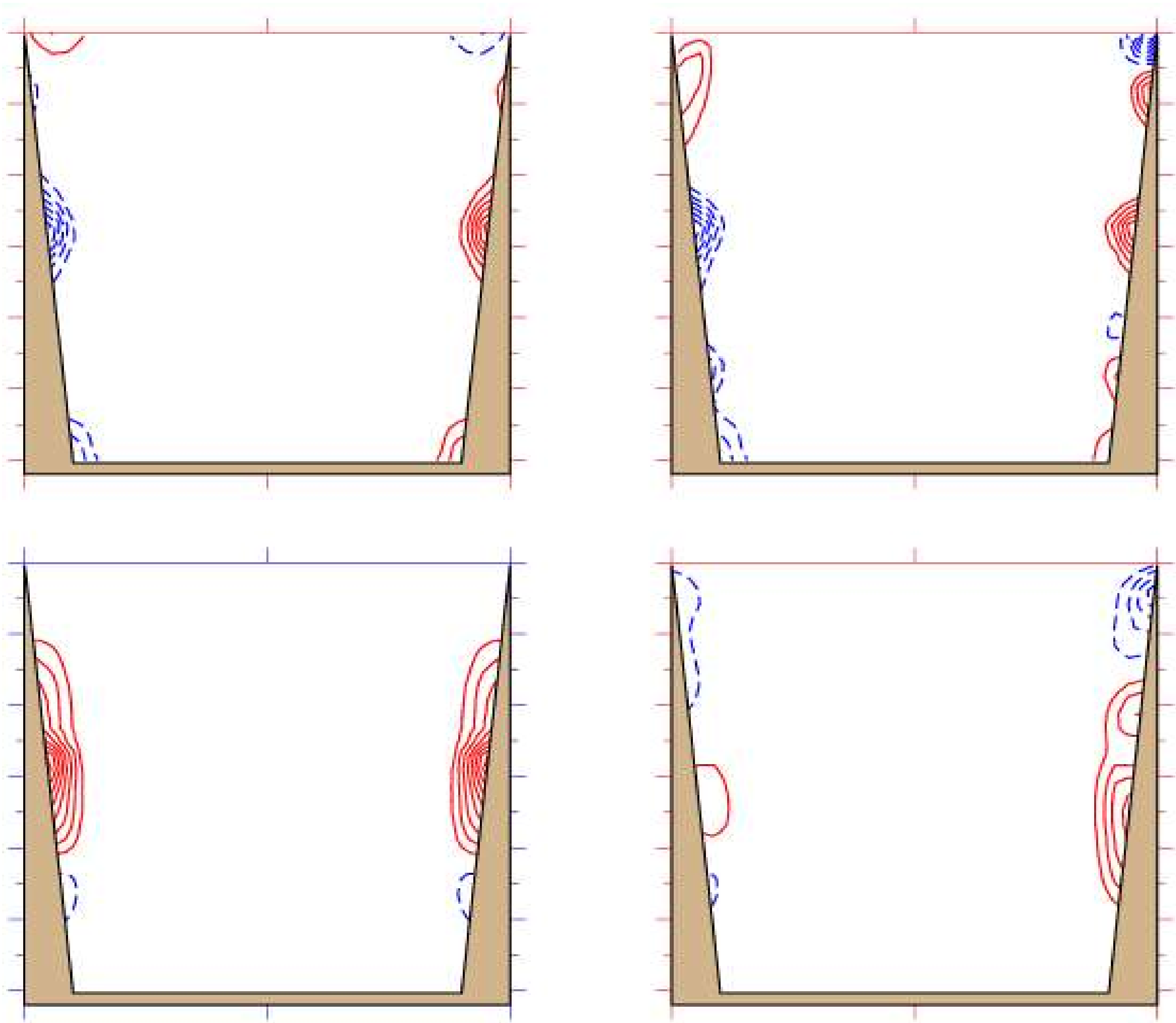
Since the thermal boundary flow is always opposite to the σ-error velocity, whether or not a model could produce the thermal boundary flow could be used to estimate the significance of the σ-error.

Background mixing coefficient $K_m = 10^{-4}$ m^2/s

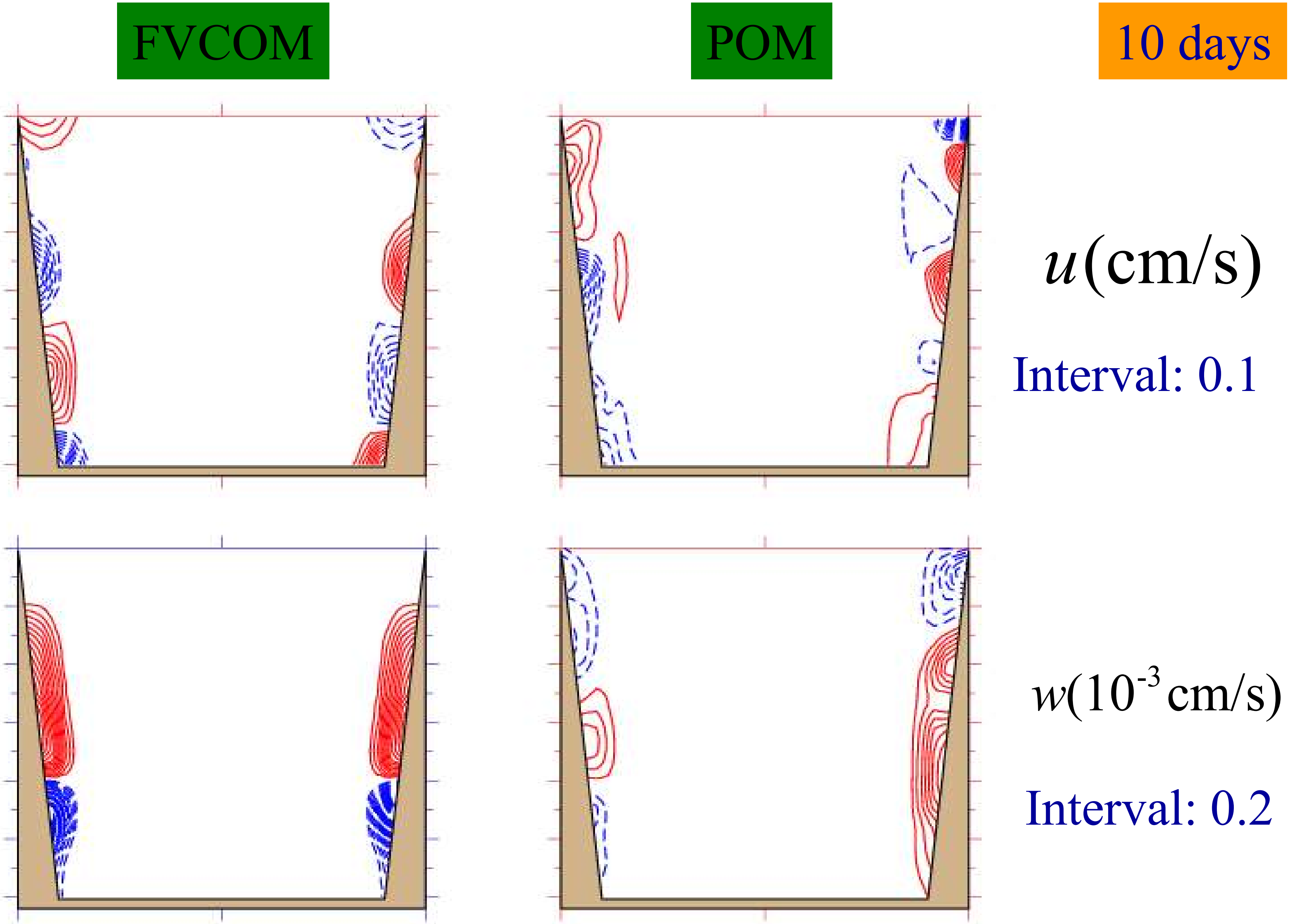


$$\alpha = \frac{300\ \text{m}}{13.5 \times 10^3\ \text{m}}$$
$$= 0.02$$

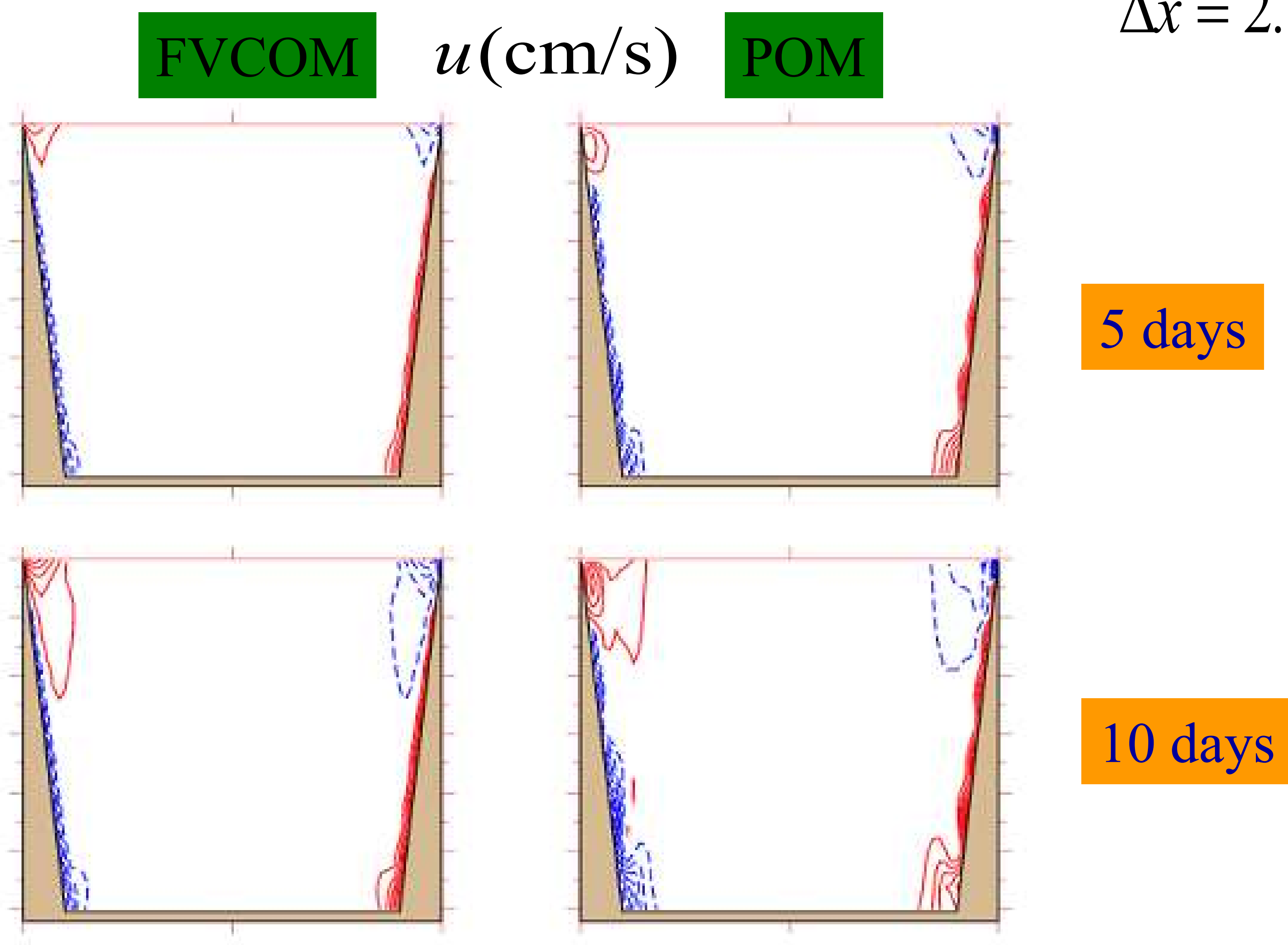
FVCOM
POM
5 days



$\Delta x = 4.5$ km
u(cm/s)
Interval: 0.1
w(10^{-3} cm/s)
Interval: 0.2


FVCOM
POM
10 days
u(cm/s)
Interval: 0.1
w(10-3 cm/s)
Interval: 0.2

$\Delta x = 2.25$ km



Recommended references

Haney, R. L. (1991), On the Pressure Gradient force over steep topography in sigma coordinate ocean models. J. Phys. Oceanogr., *21*:610-619.

Chen, C. and R. C. Beardsley (1995), A numerical study of stratified tidal rectification over finite-amplitude banks. Part I: symmetric banks. *J. Phy. Oceanogr.*, *25*, 2090-2110.