Breaking internal wave groups: Mixing and momentum fluxes

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Breaking groups of large-amplitude internal gravity waves are simulated numerically and the resulting diapycnal mixing and residual momentum are quantified. The wave frequency strongly affects the mixing, with high- and low-frequency waves doing many times more mixing than intermediate-frequency waves with the same steepness. The total residual momentum remaining in the breaking region after the remnants of the wave group propagates away shows similar frequency dependence as the diapycnal mixing. Additionally, the propagation of the breaking events and the spatial distribution of the mixing are found to agree qualitatively with a kinematic description of breaking internal wave groups. © 2011 American Institute of Physics. [doi:10.1063/1.3638155]

I. INTRODUCTION

Breaking internal waves are the main source of mixing in the ocean’s interior. Mechanisms of internal wave breaking include wave-wave interactions, impingement on critical levels, and self-acceleration. Examples of wave-wave interactions include parametric subharmonic instability [described concisely in Ref. 1] and high-mode waves impinging on critical levels created by low-mode waves. Critical levels are where the background horizontal velocity equals the horizontal phase speed of the wave.\textsuperscript{2} Self-interaction is the interaction of nonlinear internal waves with their own wave-induced mean flow.\textsuperscript{3}

Observations of internal waves and mixing on large scales in the ocean include enhanced mixing over rough topography in the Brazil Basin during the World Ocean Circulation Experiment,\textsuperscript{4} internal wave generation over the Hawaiian ridge,\textsuperscript{5} and enhanced mixing due to parametric subharmonic instability near the critical latitude.\textsuperscript{6} Mixing associated with internal waves has also been observed on smaller scales. For example, patches of elevated mixing have been observed in the vicinity of a near-inertial feature, which may indicate that the near-inertial wave is creating critical levels for smaller-scale waves.\textsuperscript{7} Overturns associated with small-scale internal wave groups appear in data collected during the Marine Boundary Layer Experiment.\textsuperscript{8}

Mixing in the ocean acts in concert with stirring to determine the distribution of ocean properties. Proposed mechanisms for small-scale mixing and stirring include stirring by geostrophically adjusted lenses of mixed fluid,\textsuperscript{9–11} ageostrophic motions with strong vertical velocities,\textsuperscript{12} and shear dispersion by mesoscale quasi-ageostrophic stirring.\textsuperscript{13} Breaking internal waves are important to both the stirring by mixed lenses hypothesis (also called the vortical mode stirring hypothesis) and the quasi-ageostrophic stirring hypothesis. Breaking internal waves are responsible for creating the mixed lenses in the vortical mode stirring hypothesis and may imbue them with momentum, such that they are not adjusting from rest. The cascade of tracer variance to small scales in the quasi-ageostrophic stirring hypothesis is arrested by vertical mixing, presumably due to breaking internal waves.

Internal waves and their contributions to momentum fluxes and mixing have been extensively studied in the atmosphere using both theory and simulations. An early study of wave-mean flow exchanges is the interaction of mountain waves with an altitude-dependent mean flow.\textsuperscript{14} The original description of critical levels using WKB theory was set in an atmospheric context.\textsuperscript{2} Later numerical simulations showed that self-acceleration allows internal gravity waves to propagate past the critical levels one would predict by WKB theory.\textsuperscript{15} Patchy wave breaking is also important in the atmosphere, as it has been shown to reduce the estimates of fluxes from what one would obtain using a homogeneous turbulent diffusivity.\textsuperscript{16}

In this paper, we describe the mixing and momentum fluxes associated with breaking internal wave groups in two dimensions and present preliminary results on the differences between breaking in two and three dimensions. Section II describes previous work on this problem and gives some theoretical background. Section III describes the numerical model used in this study, the model setup, and the initial conditions. Section IV describes the evolution of an example wave group. Section V describes the effect of wave frequency on the mixing, residual momentum, and propagation of breaking events. Section VI discusses the importance of resolution and the effects of hyperviscosity. Section VII discusses three-dimensional effects. Section VIII contains the concluding discussion.

II. THEORETICAL BACKGROUND

Internal waves in the ocean and atmosphere exhibit a wide range of behaviors, depending on the details of the system. We are especially interested in breaking internal wave groups and restrict the background presented here to results...
on stability conditions, energy budgets, and the spatial distribution of mixing in the water column.

A. Stability

Internal gravity waves are subject to various instabilities, such as parametric subharmonic instability, modulational instability, and steepening by self-acceleration. Even finite-amplitude plane internal waves by themselves are unstable, without considering the effects of a group. The most important instability for the work presented here is steepening by self-acceleration.

Self-acceleration is the interaction of a wave group with its own wave-induced mean flow, which may cause an initially statically stable wave group to steepen and break. A wave group’s susceptibility to steepening and breaking due to self-acceleration depends on its non-dimensional steepness $\epsilon$ and its frequency $\omega_0$. The non-dimensional steepness $\epsilon$ is defined as the maximum of the vertical displacement $\xi$ divided by the horizontal wavelength $\lambda_x$.

$$\epsilon \equiv \frac{\max|\xi|}{\lambda_x}.$$  (1)

The wave frequency $\omega_0$ is related to the angle of the wave vector with respect to the horizontal $\theta$ via the dispersion relation

$$\omega_0 = N_0 \cos \theta,$$  (2)

where $N_0$ is the background buoyancy frequency. A wave group will eventually steepen and break due to self-acceleration if the condition

$$\epsilon > \frac{1}{2\pi\sqrt{2}} \sin 2\theta$$  (3)

is satisfied.

A wave group is statically stable if the density $\rho$ decreases with height $z$,

$$\frac{\partial \rho}{\partial z} < 0.$$  (4)

As self-acceleration steepens the wave group, the static stability condition (4) will no longer be satisfied everywhere in the wave group. However, failure to satisfy the static stability condition does not necessarily indicate breaking. If the wave frequency is higher than the growth rate of the instability, then the wave may reverse the overturning before breaking occurs.

An approximation to the growth rate of the fastest growing instability may be derived from the Taylor-Goldstein equation by setting the background velocity to zero (e.g., set $U = 0$ in Eq. (12.58) of Ref. 20). Taking the limit $H \rightarrow \infty$ yields a growth rate equal to $\sqrt{-N^2}$, where $N$ is the local buoyancy frequency

$$N^2 = N_0^2 + \Delta N^2.$$  (5)

Thus, the condition for stability is

$$\omega_0 > \text{Re}(\sqrt{-N^2})$$  (6)

The condition (6) is called the Rayleigh-Taylor condition, and it has been shown to be a good predictor of convective overturning.

B. Energy budgets

Energy budgets are used to examine the fraction of a breaking wave group’s energy that goes to mixing, and to evaluate the performance of the numerical model. The variable names and equations in this section follow those in Ref. 21, although they have been re-written to include $\nabla^6$ hyperdissipation and hyperdiffusion. Reference 22 also contains a useful discussion of minimum potential energy in terms of the probability density function of fluid density.

The Boussinesq momentum equation with hyperviscosity $\nu'$ is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \rho + \frac{\rho}{\rho_0} \frac{\partial \mathbf{z}}{\partial t} + \nu' \nabla^6 \mathbf{u},$$  (7)

and the density equation with hyperdiffusivity $\kappa'$ is

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = \kappa' \nabla^6 \rho.$$  (8)

The total kinetic energy in the control volume $V$ under the Boussinesq approximation is

$$E_k \equiv \frac{\rho_0}{2} \int_V |\mathbf{u}|^2 \, dV.$$  (9)

The rate of change of the kinetic energy is found by taking the dot product of the momentum density $\rho_0 \mathbf{u}$ and the momentum Eq. (7) and integrating over the control volume $V$ with surface $S$

$$\frac{dE_k}{dt} = -\frac{\rho_0}{2} \oint_S |\mathbf{u}|^2 \mathbf{u} \cdot \hat{n} \, dS - \oint_S \rho_0 \mathbf{u} \cdot \hat{n} \, dS - \oint_V \rho \mathbf{w} \, dV$$

$$+ \rho_0 \nu' \oint_S \left[ (\nabla^4 \mathbf{u}) \cdot (\nabla^4 \mathbf{u}) + (\nabla^2 \mathbf{u}) \cdot (\nabla^2 \mathbf{u}) \right] \cdot \hat{n} \, dS$$

$$- \rho_0 \nu' \oint_V \left[ \nabla (\nabla^2 \mathbf{u}) \right] : \nabla (\nabla^2 \mathbf{u})^T \, dV.$$  (10)

The advection term on the right-hand side of the kinetic energy budget (10) is the advection of kinetic energy out of the control volume. The exchange term is the exchange of kinetic energy for potential energy, also called the buoyancy flux. The diffusion term is the diffusion of kinetic energy out of the control volume, and the dissipation term is the conversion of kinetic energy to heat.

The total gravitational potential energy $E_p$ in the control volume $V$ is

$$E_p \equiv g \int_V \rho z \, dV.$$  (11)
The time rate of change of the potential energy is found by multiplying the density Eq. (8) by \( g z \) and integrating over the control volume \( V \),

\[
\frac{dE_b}{dt} = -g \int_V \rho z u \cdot \hat{n} \, dS + g \int_V \rho w \, dV \\
+ g \kappa' \int_S z V [\nabla^4 \rho] \cdot \hat{n} \, dS - g \kappa' \int_V \frac{\partial}{\partial z} [\nabla^4 \rho] \, dV .
\]

(12)

The advection and diffusion terms represent the flux of potential energy out of the control volume. The exchange with kinetic energy appears with the opposite sign from the kinetic energy budget (10), and the conversion term represents the transformation of internal energy to gravitational potential energy.

The minimum potential energy \( E_b \) is defined by

\[
E_b = g \int_V \rho z_s \, dV,
\]

where \( z_s \) is the depth where the water parcel would be located if all isopycnals were horizontal. The subscript “b” is for “background.” The rate of change of the background energy \( E_b \) is found by multiplying the density Eq. (8) by \( g z_s \) and integrating over the control volume \( V \),

\[
\frac{dE_b}{dt} = -g \int_V z_s u \cdot \nabla \rho \, dV \\
+ g \kappa' \int_S z_s V [\nabla^4 \rho] \cdot \hat{n} \, dS \\
- g \kappa' \int_V V z_s \cdot \nabla [\nabla^4 \rho] \, dV .
\]

(14)

The volume integral of \( \rho \partial z_s / \partial t \) is identically zero,\(^{21} \) thus Eq. (14) is the complete background energy budget. To avoid calculating derivatives of \( z_s \) in our diagnostics, we compute the diapycnal mixing as the difference of two terms

\[
g \kappa' \int_V V z_s \cdot \nabla [\nabla^4 \rho] \, dV = -g \kappa' \int_V z_s \nabla^6 \rho \, dV \\
+ g \kappa' \int_S z_s V [\nabla^4 \rho] \cdot \hat{n} \, dS .
\]

(15)

The available potential energy \( E_a \) is the difference between the total and minimum potential energies,

\[
E_a = E_p - E_b .
\]

(16)

The available potential energy represents the potential energy that may be exchanged for kinetic energy. The available potential energy budget is the difference of the potential energy budget (12) and the background energy budget (14),

\[
\frac{dE_a}{dt} = -g \int_V (z - z_s) u \cdot \nabla \rho \, dV + g \int_V \rho w \, dV \\
+ g \kappa' \int_S (z - z_s) V [\nabla^4 \rho] \cdot \hat{n} \, dS - g \kappa' \int_V \frac{\partial}{\partial z} [\nabla^4 \rho] \, dV \\
+ g \kappa' \int_V V z_s \cdot \nabla [\nabla^4 \rho] \, dV .
\]

(17)

Energy budgets for our simulations are computed and discussed in Sec. V B.

C. Propagation of breaking events

The propagation of breaking events in an internal wave group and whether or not the breaking wave group leaves behind a continuous “scar” of mixing are vital to the stirring by mixed lenses hypothesis. For example, the spatial distribution of the breaking will determine the height \( H \) and length \( L \) of the mixed lenses. These length scales, along with the degree of mixing, will determine the velocity scale \( U \) in the lenses as they adjust. Together, these three scales will determine the Burger number \( R_u \equiv g H / f_0 L^2 \) and the Rossby number \( R_o \equiv U / f_0 L \) (Ref. 23) of the lenses.

Reference 24 describes internal wave group breaking as occurring when and where a quantity (e.g., Richardson number) exceeds a critical value. For a quantity to exceed the critical value at some location, the phase at that location must be within a certain range, and the location must be close enough to the center of the group such that the amplitude of the quantity is sufficiently large. Lines of constant phase move with the phase velocity and the amplitude envelope moves with the group velocity. Thus, Ref. 24 predicts that the velocity of the breaking events \( \mathbf{c}_b \) is the vector sum of the group velocity \( \mathbf{c}_g \) and the phase velocity \( \mathbf{c}_p \) of the fundamental wave vector \( \mathbf{k}_0 \),

\[
\mathbf{c}_b = \mathbf{c}_g + \mathbf{c}_p .
\]

(18)

The prediction (18) in a non-rotating system is simply

\[
\mathbf{c}_b = \left( \frac{N_0}{|k_0|}, 0 \right) ,
\]

and we evaluate this prediction in Sec. V C.

III. NUMERICAL MODEL

We use a modified version of the model described in Ref. 25 to simulate breaking internal wave groups in a continuously stratified fluid. The model equations are nonlinear, Boussinesq, and include hyperviscosity \( \nu' \) and hyperdiffusivity \( \kappa' \). The model pressure \( p(x, t) \) is decomposed into background \( \bar{p}(z) \) and perturbation \( p'(x, t) \) values,

\[
p(x, t) = \bar{p}(z) + p'(x, t) .
\]

(20)

Similarly, the model density is decomposed into an average \( \rho_0 \), background \( \bar{\rho}(z) \), and perturbation \( \rho'(x, t) \) value,
\[ \rho(x, t) = \rho_0 + \tilde{\rho}(z) + \rho'(x, t). \]  

(21)

The decompositions in Eqs. (20) and (21) allow the elimination of \( \tilde{\rho}(z) \) from the equations of motion, yielding the model momentum equation

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\mathbf{V}\rho'}{\rho_0} - g \frac{\rho'}{\rho_0} \frac{\partial z}{w} + \mathbf{v}' \nabla^6 \mathbf{u}. \]  

(22)

The model equation for the density perturbation \( \rho'(x, t) \) is

\[ \frac{\partial \rho'}{\partial t} + (\mathbf{u} \cdot \nabla) \rho' = -w \frac{d\tilde{\rho}}{dz} + \kappa' \nabla^6 (\rho' + \tilde{\rho}). \]  

(23)

Finally, the model forces the velocity field to be non-divergent

\[ \nabla \cdot \mathbf{u} = 0. \]  

(24)

The model originally used hyperviscosity and hyperdiffusion of the form \( \mathbf{v}'(\partial_x^6 + \partial_y^6 + \partial_z^6) \mathbf{u} \) and \( \kappa' (\partial_x^6 + \partial_y^6 + \partial_z^6) \rho' \) in the last terms of Eqs. (22) and (23). We use a modified version of the model with full \( \nabla^6 \) hyperviscosity and hyperdiffusion in Eqs. (22) and (23), so that dissipation is isotropic. The model with full \( \nabla^6 \) hyperviscosity and hyperdiffusion dissipates energy slightly faster than the original model.

Time integration of the model equations is done by operator splitting using alternating explicit and implicit steps. The explicit step uses third-order Adams-Bashforth to integrate the advection and gravity terms. The implicit step uses second-order Adams-Moulton to integrate the hyperviscosity and hyperdiffusion terms. The pressure perturbation \( \rho'(x, t) \) is computed by enforcing non-divergence of the velocity field (24).

The model domain is periodic in all directions, allowing fast and accurate calculations of spatial derivatives via Fourier transforms. However, the periodicity also allows waves to re-enter the domain after propagating through the boundaries.

Sponge layers are used to eliminate the waves that would otherwise re-enter the domain after propagating through the upper and lower boundaries. The sponge layers exponentially damp the density anomaly \( \rho' \) and velocity \( \mathbf{u} \) with decay rate \( r \). The sponge layers work by adding an extra term to the right-hand sides of the momentum and density Eqs. (22) and (23). For example, the model density Eq. (23) becomes

\[ \frac{\partial \rho'}{\partial t} + (\mathbf{u} \cdot \nabla) \rho' = -w \frac{d\tilde{\rho}}{dz} + \kappa' \nabla^6 (\rho' + \tilde{\rho}) - r(z) \rho'. \]  

(25)

The spatial dependence of the decay rate \( r \) is Gaussian in \( z \),

\[ r(z) = \left( e^{-z^2/L_{sponge}^2} + e^{-(z-L_z)^2/L_{sponge}^2} \right)/\tau_{sponge}, \]  

(26)

where \( L_{sponge} \) is the length scale of the sponge layers, \( \tau_{sponge} \) is the damping time scale, and \( L_z \) is the height of the domain. We do not use sponge layers at the left and right edges of the domain because they would disrupt the horizontal wave-induced mean flow.

A. Model setup

We are interested in wave groups that are modulated in the vertical and horizontal directions, and so we require a model domain that is much larger than a wave group in order to allow the wave group space to propagate, steepen, and break before encountering a sponge layer. Our model domain is a square 300 m on a side and used to accommodate wave groups approximately 60 m in diameter. The model grid is uniform in \( x \) and \( z \) with \( 2304 \times 2304 \) grid points and 0.13 m spacing.

The background buoyancy frequency \( N_0 \) is chosen to be constant with \( N_0 = 1/150 \) s\(^{-1}\), which corresponds to a buoyancy period \( T = 2\pi/N_0 \) of 15.7 min. The background density gradient \( d\tilde{\rho}/dz \) may be computed from the equation for the squared buoyancy frequency,

\[ N_0^2 = -\frac{g}{\rho_0} \frac{d\tilde{\rho}}{dz}. \]  

(27)

Taking \( g = 9.81 \) m s\(^{-2}\) and \( \rho_0 = 1024 \) kg m\(^{-3}\), the background density gradient is \( d\tilde{\rho}/dz = -4.6 \times 10^{-3} \) kg m\(^{-4}\).

The hyperviscosity \( \nu' \) and hyperdiffusivity \( \kappa' \) are both \( 10^{-10} \) m\(^5\) s\(^{-1}\), yielding a Prandtl number \( \nu'/\kappa' \) of one. \( \nu' \) and \( \kappa' \) are sufficiently small such that buoyancy effects dominate viscous effects at the energy-containing scales.

Finally, the parameters of the sponge layer are \( L_{sponge} = 5 \) m and \( \tau_{sponge} = 500 \) s for the layer length scale and decay time scale (26). We omit the upper and lower 25 m (192 grid points) of the domain from our analysis (Fig. 1). Inside the remaining domain, the sponge layer damping rate \( r \) is sufficiently small such that \( 1/r \geq 3.8 \times 10^{10} \) buoyancy periods.

B. Initial conditions

The initial condition is a plane wave modulated by a horizontally and vertically localized Gaussian envelope [e.g., as in Ref. 19]. The initial conditions of all dependent variables are completely specified by the streamfunction \( \psi_0(x) \),

\[ \psi_0(x, z) = \frac{2\pi N_0 \epsilon}{k_0 \sqrt{k_0^2 + m_0^2} + m_0^2} \exp \left[ -\frac{(x-x_0)^2 + (z-z_0)^2}{2\sigma^2} \right] \cos[k_0(x-x_0) + m_0(z-z_0)], \]  

(28)

where \( \sigma \) is the width of the envelope and the wave group is centered at \((x_0, z_0)\) in the \(xz\)-plane and at \((k_0, m_0)\) in Fourier space. The corresponding fundamental frequency \( \omega_0 \) according to the linear dispersion relation is

\[ \omega_0^2 = \frac{N_0^2 k_0^2}{k_0^2 + m_0^2}. \]  

(29)

The initial density anomaly \( \rho'(x, 0) \) is computed from the streamfunction (28) using the relation

\[ \rho'|_{z=0} = \frac{k_0}{\omega_0} \frac{d\tilde{\rho}}{dz} \psi_0(x, z), \]  

(30)
and the corresponding initial vertical displacement $\xi$ is

$$\xi|_{t=0} = -\frac{k_0}{\omega_0} \psi_0(x, z).$$  \hspace{1cm} (31)

The raw initial velocities are the partial derivatives of the streamfunction,

$$(u, w)|_{t=0} = \left( -\frac{\partial \psi_0}{\partial z}, \frac{\partial \psi_0}{\partial x} \right).$$  \hspace{1cm} (32)

The raw velocities (32) must be corrected for the wave-induced mean flow, otherwise the wave group will leave behind a mean horizontal velocity with the same sign as the horizontal wavenumber $k_0$. This is inconsistent with our picture of a wave group propagating through an initially quiescent fluid, where the starting position is not special. Therefore, we follow a similar procedure as in Ref. 26 and modify the initial velocity field so that the total horizontal momentum of the wave group times the vertical component of the group velocity is equal to the net momentum flux of the wave group. This is equivalent to adding an approximation of the pseudomomentum to the velocity at every point.

The small-amplitude approximation to the pseudomomentum corresponding to the $x$-component of the momentum (also called the wave activity $A$) under the Boussinesq approximation in a system with zero background flow is

$$A = -\frac{g}{N^2} \rho' \zeta,$$  \hspace{1cm} (33)

where $\zeta$ is the $y$-component of the vorticity.27 The $y$-component of the vorticity is computed directly from the streamfunction,

$$\zeta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\nabla^2 \psi_0.$$  \hspace{1cm} (34)

Dividing the pseudomomentum (33) by the density $\rho_0$ and averaging over a wavelength $\lambda_x$ in the horizontal yields a velocity field

$$u_M(x) = -\frac{1}{\lambda_x} \int_{x-\lambda_x/2}^{x+\lambda_x/2} \xi_i dx,$$  \hspace{1cm} (35)

where $\xi$ is the vertical displacement. The domain average of $u_M$ is approximately equal to the wave-induced momentum flux of the wave group divided by the vertical component of the group velocity,

$$c_{xg} \langle u_M \rangle \approx \langle u w \rangle.$$  \hspace{1cm} (36)

$c_{xg}$ is the vertical component of the group velocity corresponding to the central wavenumbers,

$$c_{xg} = \begin{cases} \frac{N_0 m_0^2}{(k_0^2 + m_0^2)^{3/2}}, & k_0 > 0, \\ \frac{-N_0 m_0^2}{(k_0^2 + m_0^2)^{3/2}}, & k_0 < 0. \end{cases}$$  \hspace{1cm} (37)

The domain average $\langle f(x) \rangle$ in Eq. (36) is defined by

$$\langle f(x) \rangle = \frac{1}{V} \int_V f(x) \, dx.$$  \hspace{1cm} (38)

The approximation (36) becomes exact in the limit that the number of waves in the group $\sigma k_0$ goes to infinity. The approximation in Eq. (36) is good to within $1\%$ for the initial conditions used in this study.

To include the velocity $u_M$ while maintaining continuity (24), we add $\partial u_M/\partial z$ to the vorticity given by $-\nabla^2 \psi$ and find the Laplacian of the total streamfunction $\psi_T$,

$$-\nabla^2 \psi_T = -\nabla^2 \psi_0 + \frac{\partial u_M}{\partial z}.$$  \hspace{1cm} (39)

Inverting Eq. (39) yields the total streamfunction.

The partial derivatives of the total streamfunction give the corrected initial velocity field $(u_T, w_T)$. The net momentum flux of the corrected velocities $\langle u_T w_T \rangle$ agrees with the momentum flux of the raw velocities to within less than $1\%$. The original streamfunction $\psi_0$ (28) is used to find the density anomaly in Eq. (30), and the wave-induced mean flow computed using the corrected vorticity $\zeta_T$ agrees with the wave-induced mean flow computed using the raw vorticity to within less than $1\%$. Thus, we have applied a correction that preserves the net momentum flux and pseudomomentum of the wave group, but sets the net momentum times the

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FIG. 1. The initial conditions specified by the streamfunction $\psi$ (Eq. (28)) for a wave group with steepness $\epsilon = 0.14$, central wavenumbers $k_0 = m_0 = \pi/5$ rad m$^{-1}$, and width $\sigma = 10\sqrt{2}$ m. (a) The Gaussian envelope that modulates the plane wave. The contours are at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ times the maximum value of the envelope. The hatched regions at the top and bottom of the domain contain sponge layers and are excluded from the analysis. (b)–(d) Close-ups of the initial density anomaly $\rho'$, horizontal velocity $u$, and vertical velocity $w$, respectively. The solid contours are where each variable equals $\frac{1}{4}$ and $\frac{3}{4}$ of its maximum value, and the dotted contours are where each variable equals $\frac{1}{2}$ and $\frac{1}{2}$ of its maximum value. The velocities in (c) and (d) are computed from the total stream function $\psi_T$ (Eq. (39)).
vertical component of the group velocity equal to the net momentum flux. Plane waves do not require the correction (39), because plane waves have a non-divergent momentum flux.

The parameter values for our reference wave group simulation are steepness \( \varepsilon = 0.14 \), central wavenumbers \( k_0 = m_0 = \pi/5 \text{ rad m}^{-1} \), width \( \sigma = 10\sqrt{2} \text{ m} \), and center \( (x_0, z_0) = (75 \text{ m}, 225 \text{ m}) \). The angle of propagation of phase with respect to the horizontal is \( \theta = 45^\circ \). The correction for the wave-induced mean flow (39) can be seen in the asymmetry between the positive and negative values of \( u \) and \( w \) (Fig. 1).

IV. WAVE GROUP EVOLUTION

The base case simulation (henceforth referred to as the reference wave group; Fig. 1) is initially statically stable with an amplitude about 85% of that required for overturning but is unstable to self-acceleration (Fig. 2). The wave group will break regardless of whether it is initialized with the raw streamfunction \( \psi_0 \) (28) or the corrected streamfunction \( \psi_T \) (39). We use the corrected streamfunction because it results in cleaner velocity profiles.

The density perturbation is initially symmetric, but becomes asymmetric by \( t = 4 \) buoyancy periods, with the trailing edge narrowing and the leading edge broadening (Fig. 3). By \( t/T = 8 \), the trailing edge starts to become ragged. This is consistent with theoretical work showing that for \( |\theta| < \arcsin \sqrt{3}/5 \approx 50.8^\circ \), the amplitude of the wave group grows on the trailing edge. The wave steepens and begins breaking between \( t/T = 12 \) and 16. Following the breaking, energy radiates away from the breaking region, with the majority of the remaining wave energy propagating downward until it is absorbed by the sponge layer (not shown). The turbulent breaking enhances mixing in the breaking region and leaves behind a residual momentum. By \( t/T = 50 \), there is separation between the remnants of the wave group propagating out the top and bottom of the domain and the residual momentum in the breaking region (Fig. 4).

FIG. 2. Thresholds of stability in the frequency-steepness plane. The angle \( \theta = \arccos (\alpha/N_0) \) corresponds to the wave frequency. The non-dimensional wave steepness \( \varepsilon \) equals the maximum vertical displacement divided by the horizontal wavelength (1). The circles indicate the locations of the simulations in this study and the filled circle is the reference simulation (Fig. 1). The curves correspond to the thresholds for self-acceleration \( \varepsilon_{sa} \), overturning \( \varepsilon_{ot} \), and convective instability \( \varepsilon_{cy} \) [after Fig. 3 of Ref. 19].

FIG. 3. Contours of the absolute value of the density anomaly \( \rho' \) at six different times for the initial condition in Fig. 1. Time \( t/T \) is in buoyancy periods, where \( T = 2\pi/N_0 \). The plotting domain moves with the group velocity corresponding to the central wavenumbers \( k_0 \) and \( m_0 \) (Eq. (37)). The black contours are at \(|\rho' (x,t)| = 1/4 \) and \( 1/4 \times \max(|\rho' (x,0)|) \) and the gray contours represent the initial condition. The steepness of the wave group is \( \varepsilon = 0.14 \), the central wavenumbers are \( k_0 = m_0 = \pi/5 \text{ rad m}^{-1} \), and the width is \( \sigma = 10\sqrt{2} \text{ m} \).
The momentum flux associated with a wave group is divergent, unlike the non-divergent momentum flux associated with a plane wave. The momentum flux is reduced by wave breaking, which produces sharp gradients and allows dissipation to reduce the correlation $h_{uw}$. Thus, the momentum flux is convergent where breaking occurs, and a residual horizontal velocity is found in the breaking region after the remainder of the group propagates away. If the wave group does not break, then there is negligible residual momentum, as shown by a comparison between the reference wave group and a wave group that is identical to the reference wave group, except that its amplitude is smaller. $T = 2\pi/N_0$ is the buoyancy period, and the vertical profiles are offset by 0.025 cm s$^{-1}$ for clarity. The central wavenumbers of each wave group are $k_0 = m_0 = \pi/5$ rad m$^{-1}$, and the width is $\sigma = 10\sqrt{2}$ m.

![FIG. 4. Vertical profiles of horizontally averaged horizontal velocities $\bar{u}(z,t)$ for a breaking wave group ($\epsilon = 0.14$, solid line) and a nonbreaking wave group ($\epsilon = 0.12$, dashed line). The breaking wave group is the reference wave group (Fig. 1). The non-breaking wave group is identical to the reference wave group, except that its amplitude is smaller. $T = 2\pi/N_0$ is the buoyancy period, and the vertical profiles are offset by 0.025 cm s$^{-1}$ for clarity. The central wavenumbers of each wave group are $k_0 = m_0 = \pi/5$ rad m$^{-1}$, and the width is $\sigma = 10\sqrt{2}$ m.](image)

One of the motivations for this work is to study the spatial distribution of mixing. In a non-rotating system, mixed fluid spreads out along isopycnals, as there is no Coriolis effect to balance the pressure gradient force. Thus, there are no obvious mixed patches in the evolution of the wavepacket (Fig. 3). To see which fluid mixes, we integrate the dissipation of density variance at each point in the domain. The

![FIG. 5. The change in kinetic energy $\Delta E_k$ in the control volume $z \in [25\text{ m}, 275\text{ m}]$ due to the time integral of each of the terms on the right-hand side of the kinetic energy budget (10) for the wave group in Fig. 1. The changes in energy are computed from $t/T = 1$ to exclude the initial oscillations in the model. The “Model” line is the actual change in kinetic energy in the model and the “Check” line is the sum of the terms on the right-hand side of the kinetic energy budget (10). The figure is cut off at $t/T = 45$, because this is when the kinetic energy begins to drop due to the wave group propagating out of the control volume.](image)

![FIG. 6. The change in available potential energy $E_a$ in the control volume $z \in [25\text{ m}, 275\text{ m}]$ due to the time integral of each of the significant terms on the right-hand side of the available potential energy budget (17). The changes in energy are computed from $t/T = 1$ to exclude the initial oscillations in the model. The “Model” line is the actual change in available potential energy in the model and the “Check” line is the sum of all of the terms on the right-hand side of the available potential energy budget (17). The figure is cutoff at $t/T = 45$, because this is when the wave group begins propagating out of the control volume.](image)
density variance budget is obtained by multiplying the density Equation (23) by \( \rho' \),

\[
\frac{\partial (\rho'^2)}{\partial t} = - \nabla \cdot (u \rho'^2) - 2w \rho' \frac{d \rho}{dz} + 2k' \nabla \cdot (\rho' \chi_5 \rho' - \nabla^2 \rho' \nabla \rho') + \nabla^2 (\rho' \nabla \rho') - 2k' |\nabla^3 \rho'|^2. \tag{40}
\]

The last term on the right-hand side of the density variance budget is the dissipation of density variance. We see clearly that the greatest dissipation of density variance occurs in the path of the wave group and to the right of the path (Fig. 7). This is consistent with the predicted propagation of the breaking region (19).

V. FREQUENCY DEPENDENCE

We vary the fundamental frequency by changing only the vertical wave number \( m_z \), and so hold the wave steepness \( \epsilon \), maximum vertical displacement, and initial potential energy constant. The initial kinetic energy varies only due to the correction for the wave-induced mean flow (39). The initial conditions we consider span wave groups that are unstable by three different criteria: the self-acceleration criterion, the overturning criterion, and the Rayleigh-Taylor criterion (Fig. 2, after Fig. 3 of Ref. 19).

A. Dependence of residual momentum on frequency

Momentum density in the Boussinesq approximation is exactly \( \rho_0 u \) (neither \( \rho(z) \) nor \( \rho'(x, t) \) appear). Thus, it suffices to consider \( u \) alone, and in this section, we will use the term momentum to mean the velocity. We define the total residual momentum \( U_{\text{res}} \) to be

\[
U_{\text{res}} = \int_{z_1}^{z_2} u(z) \, dz, \tag{41}
\]

where \( z_1 \) and \( z_2 \) are the upper and lower boundaries of the breaking region. The operational definition of the breaking region is the range of \( z \)-values in which the Rayleigh-Taylor condition (6) is satisfied at some time during the simulation (Fig. 7). The time-integrated dissipation of density variance (last term on the right-hand side of Eq. (40)) at each point in the domain in the reference simulation (Fig. 1). 50% of the total dissipation of density variance occurs in the black shaded region and 95% in the light gray region. The dashed circle has radius \( \sigma \) and is centered at \((z_0, 0)\). The dashed lines are parallel to the group velocity corresponding to the central wavenumbers of the group (Eq. (37)).

The operational definition of the breaking region is the range of \( z \)-values in which the Rayleigh-Taylor criterion, and the Rayleigh-Taylor criterion by three different criteria: the self-acceleration criterion, and convective instability \( \theta_{av} \) by waves with steepness \( \epsilon = 0.14 \) are indicated by vertical dashed lines. (c) \( U_{\text{res}}/U_m \) as a function of angle for the same simulations as in (b).

B. Dependence of mixing on frequency

In general, the onset of elevated mixing coincides with the onset of breaking and occurs earliest for high frequency wave groups (small \( \theta \) and latest for intermediate frequency wave groups. The rate of mixing when the waves are overturning, but not yet breaking, is much lower than during the breaking. The rate of mixing as a function of time is asymmetrical, with a sharp increase in mixing at the beginning of the breaking and a gradual decrease after the end of the breaking (Fig. 9).

The mixing as a function of \( \theta \) (Fig. 9) shows a similar pattern to the residual momentum, with minimal mixing occurring for intermediate frequencies (near \( \theta = 45^\circ \)), and the highest mixing occurring for high frequency waves (small \( \theta \)). There are no obvious changes in the mixing across the critical angle for overturning \( \theta_{av} \); and there is a large
increase in mixing across the critical angle for convective instability $\theta_{cv}$ (Fig. 9).

C. Propagation of breaking events

The propagation of breaking events, and whether or not breaking events overlap, determines the size and shape of the resulting mixed patch. These quantities are especially important to the stirring by mixed lenses hypothesis. Reference 24 predicts that the velocity of breaking events will be the vector sum of the group and phase velocities (18), which is horizontal in a non-rotating reference frame (19). We evaluate this prediction for a non-rotating frame (19) using two different operational definitions of the breaking events. The first operational definition of a breaking event is a connected set of points that violate the static stability condition (4). The second operational definition of a breaking event is a connected set of points that violate the Rayleigh-Taylor condition (6). The points violating the Rayleigh-Taylor condition are a subset of those violating the overturning condition (4).

The prediction that the horizontal speed of breaking events will be $N_0 = j k_0 j$ (Eq. (19)) agrees with the early parts of the simulations for $\theta = 30^\circ$, $45^\circ$, and $60^\circ$ regardless of whether the overturning condition or the Rayleigh-Taylor condition is used to identify breaking events. Later in the simulations, when the wave breaking has exited waves of different frequencies, there are overturns propagating at different speeds (Figs. 10(a)–10(c)).

The prediction that the vertical speed of breaking events will be zero (Eq. (19)) agrees very well in the case $\theta = 60^\circ$ using the Rayleigh-Taylor condition. The agreement for the cases $\theta = 30^\circ$ and $45^\circ$ is not as good. However, the vertical velocities of the breaking events are much smaller than the horizontal velocities. Individual overturns may span a few meters in the vertical in their lifetimes while traveling a few tens of meters in the horizontal (Fig. 10).

The leading edge of the envelope encompassing the breaking events travels at the group speed. This is most apparent in the $\theta = 45^\circ$ and $60^\circ$ simulations using the Rayleigh-Taylor condition to identify breaking events (Fig. 11).
In summary, the prediction that breaking events propagate and the sum of the group and phase velocities does well for the horizontal speed of breaking events, but not as well for the vertical speed. The leading edge of the envelope of breaking events travels with the group velocity corresponding to the fundamental wave vector of the group.

VI. RESOLUTION AND HYPERVISCOSITY

Any computer model that is not a direct numerical simulation (DNS) requires a closure or filter to prevent the cascade of energy to unresolved small scales and model instability. A DNS would be too computationally intensive for modeling breaking wave groups, and so we are forced to use a closure. We use hyperviscosity in this study because it is simple and fast. The simplicity of hyperviscosity means that the mixing is straightforward to calculate (Eq. (17)). However, care must be taken to prevent the hyperviscosity from affecting the evolution of the larger scales. Then the mixing done by the hyperviscosity in the different model runs is a measure of the amount of small scales created by the different breaking wave groups.

To ensure that the hyperviscosity does not affect the larger scale motions, we performed calculations similar to Winters and D’Asaro\textsuperscript{29}, who used $\nabla^6$ hyperviscosity in a model of internal wave breaking at a critical level and Waite and Bartello\textsuperscript{30}, who used hyperviscosity in a study of turbulence generated by internal gravity waves. The conditions the model must satisfy are that there must be a clear separation between the dissipation scale and the energy containing scales and the dissipation scale must be resolved.

Reference\textsuperscript{29} plots the spectrum of vertical displacements $\zeta$ in energy-preserving form and show that the vast majority of the variance is contained in the scales where the buoyancy time scale is much smaller than the hyperviscous time scale. The same is true in the model runs presented here (Fig. 12). Furthermore, the buoyancy time scale is also much smaller than the viscous time scale one would obtain using a Laplacian viscosity with $\nu = 10^{-6}$ m$^2$ s$^{-1}$.

In the model results presented in this work, the majority of the energy is contained in the wave group, and so we also check that the larger turbulent overturns are not affected by the hyperviscosity. Reference\textsuperscript{30} shows that the turbulent

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig12.png}
\caption{The normalized spectrum of variance of vertical displacements for the model whose initial conditions are shown in Fig. 1. The spectrum is plotted in variance-preserving form and at the height of the breaking (the time of maximum energy dissipation). The horizontal wavenumber $k_*$ used for computing the viscous and hyperviscous time scales is taken to be 1 m$^{-1}$.}
\end{figure}
The non-dimensional steepness $\epsilon$ (Eq. (1)) is still taken to be 0.14. The breaking is more energetic than in the case of vertically and horizontally modulated wave groups, and so the grid spacing is decreased to 0.1 m and the hyperviscosity is increased to $\nu^\prime = 1 \times 10^{-9}$ m$^6$ s$^{-1}$.

The angle $\theta$ (Eq. (2)) is fixed at 45° to examine the effects of changing the number of dimensions from two to three. The two cases leave behind nearly identical profiles of residual momentum, although the additional averaging in the three-dimensional case makes the profile of horizontally averaged horizontal momentum smoother in the three-dimensional case (Fig. 13(b)).

The total loss of mechanical energy due to mixing and dissipation is similar in the two- and three-dimensional cases (Fig. 14). However, the energy loss is achieved by different processes. The three-dimensional instabilities grow faster
than the two-dimensional ones, resulting in initially greater mixing and dissipation of kinetic energy. However, the spanwise vortices created in the three-dimensional case dissipate kinetic energy faster than they cause diapycnal mixing. By $t/T = 8$, the mixing in the two-dimensional case has caught up with and exceeded the mixing in the three-dimensional case. By the end of the simulations, nearly the same amount of mechanical energy has been dissipated but by different processes. In the two-dimensional case, the majority of the initial mechanical energy has gone to diapycnal mixing, while in the three-dimensional case, the majority of the initial energy has been lost to kinetic energy dissipation (Fig. 14). In this model geometry, with the wave energy confined in the horizontal, the frequency dependence of the mixing and residual momentum is much weaker than in the case of the horizontally modulated wave groups.

VIII. DISCUSSION

Understanding internal wave groups is important for understanding and parameterizing mixing and stirring in the ocean. It is important to quantify the residual momentum left behind by a breaking wave group as well as the total mixing and the spatial distribution of the mixing.

The residual momentum left behind by a breaking wave group may accelerate the mean flow or affect the geostrophic adjustment of the fluid mixed by the breaking wave, whereas nonbreaking wave groups leave behind minimal residual momentum. The residual momentum left behind by a breaking wave group is strongly frequency-dependent, high- and low-frequency wave groups leave behind much more residual momentum than intermediate frequency groups of the same steepness (Fig. 8). We speculate that this may be related to the waves with $\theta = 45^\circ$ being closest to the self-acceleration threshold in the frequency-steepness plane.

The diapycnal mixing as a function of frequency follows a similar pattern as the residual momentum, with minimal mixing done by the wave groups of intermediate frequency (Fig. 9). Negligible mixing is done before the onset of wave breaking or by non-breaking waves.

The dissipation of density variance occurs in the path of the wave groups and to the side of the path corresponding to the direction of the wave-induced mean flow. Not surprisingly, the region in which more than 50% of the density variance dissipation occurs coincides with where the overturns occur.

Breaking events identified by the Rayleigh-Taylor condition (Eq. (6)) propagate in the horizontal with the speed predicted by Ref. 24. However, the vertical speed of the breakers is non-zero, in disagreement with the prediction. An alternative diagnostic of the breaking region might show better agreement. The leading edge of the envelope propagates with the group speed of the wave group.

Breaking of large-amplitude internal waves is a three-dimensional process. Theoretical work\textsuperscript{16} shows that for statically unstable plane waves, the fastest growing instabilities are three-dimensional. Numerical simulations of internal waves breaking at a critical level\textsuperscript{29} and of breaking large-amplitude inertia-gravity waves\textsuperscript{34,35} also shows the importance of the third dimension. Preliminary simulations with a slightly different model set-up indicate that the residual momentum is the same in two and three dimensions, as is the total dissipation of mechanical energy. However, the breaking is completed faster in three dimensions and with more dissipation of kinetic energy and less diapycnal mixing (Fig. 14).

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\textsuperscript{5}The Hawaiian Ocean Mixing Experiment was a multi-year field program dedicated to studying the internal waves generated by the internal tide flowing over the Hawaiian ridge. Vol. 36 Issue 6, 2006 of the Journal of Physical Oceanography is a special issue devoted to the Hawaiian Ocean Mixing Experiment.


