Linear Algebra and Calculus Review

Nomenclature:

scalar: A scalar is a variable that only has magnitude, e.g., a speed of 40 km/h, 10, a, (42 + 7), π, $\log_{10}(a)$

vector: A geometric entity with both length and direction; a quantity comprising both magnitude and direction, e.g., a velocity of 40 km/h north, velocity $\vec{u}$, position $\boldsymbol{x} = (x, y, z)$

array: An indexed set or group of elements, also can be used to represent vectors, e.g.,

row vector/array: $\begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 \end{bmatrix}$ column vector/array: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ 21 \end{bmatrix}$

matrix: A rectangular table of elements (or entries), which may be numbers or, more generally, any abstract quantities that can be added and multiplied; effectively a generalized array or vector - a collection of numbers ordered by rows and columns.

[2 x 3] matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ [m x n] matrix: $\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix}$

Examples (special matrices):

A **square matrix** has as many rows as it has columns. Matrix **A** is square but matrix **B** is not:

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 12 & 5 \\ -1 & 7 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 12 & 5 \end{bmatrix}$$

A **symmetric matrix is** a square matrix in which $x_{ij} = x_{ji}$, for all *i* and *j*. A symmetric matrix is equal to

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 12 & 10 \\ -1 & 10 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 10 & 12 & 2 \\ -1 & 10 & 0 \end{bmatrix}$$

A **diagonal matrix** is a symmetric matrix where all the off diagonal elements are 0. The matrix **D** is

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

An **identity matrix** is a diagonal matrix with only 1's on the diagonal. For any square matrix, **A**, the

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{IA} = \mathbf{A}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 12 & 10 \\ -1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 12 & 10 \\ -1 & 10 & 0 \end{bmatrix}$$

Example (system of equations):

Suppose we have a series of measurements of stream discharge and stage, measured at *n* different times.

time (day) = [0 14 28 42 56 70]

stage (m) = [0.612 0.647 0.580 0.629 0.688 0.583]

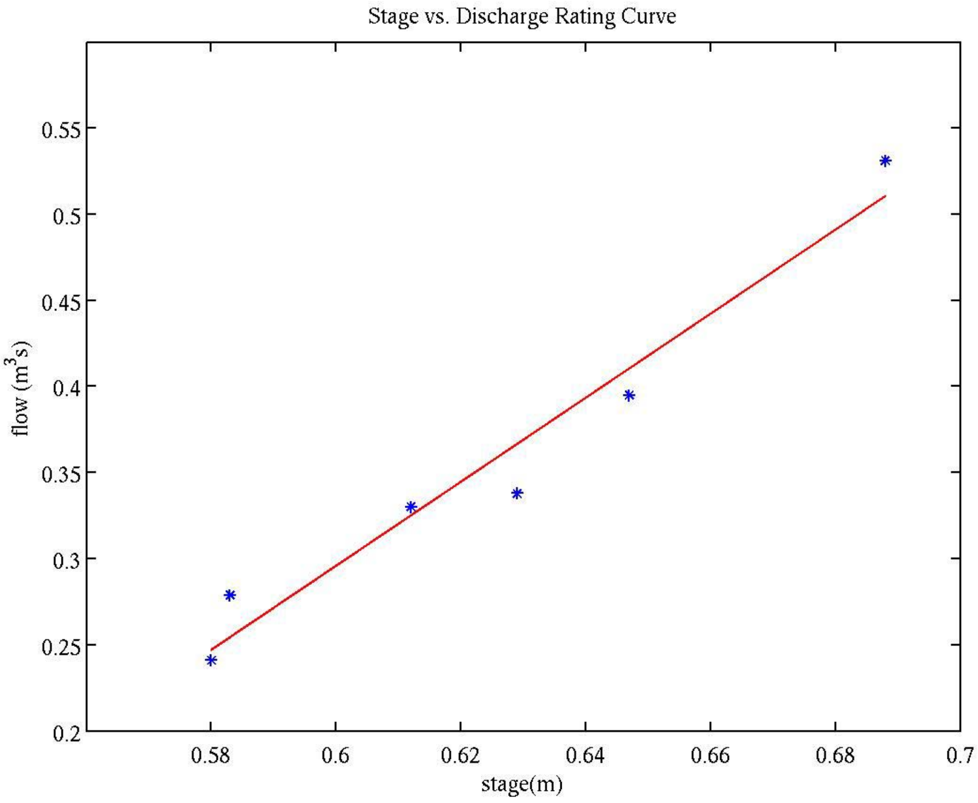
discharge (m^3/s) = [0.330 0.395 0.241 0.338 0.531 0.279]

Suppose we now wish to fit a rating curve to these measurements. Let x = stage, y = discharge, then we can write this series of measurements as: $y_i = mx_i + b$ with $i = 1{:}n$.

This in turn can be written as: $\boldsymbol{y} = \boldsymbol{X}\,\boldsymbol{b}$ →

$$\begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

$$[n \times 1] = \quad [n \times 2] \quad [2 \times 1]$$



Vectors:

Addition/Subtraction: Two vectors can be added/subtracted if and only if they are of the same dimension.

$$\mathbf{A} \pm \mathbf{B} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \pm \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2+5 \\ 3-2 \\ 4+3 \\ 1+7 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ 8 \end{bmatrix}$$

Scalar Multiplication: If k is a scalar and $\mathbf{A}$ is a vector of length n, then

$$k\mathbf{A} = k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$$

Example:

$$2\begin{bmatrix} 25 \\ 20 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\times 25 \\ 2\times 20 \\ 2\times 5 \end{bmatrix} = \begin{bmatrix} 50 \\ 40 \\ 10 \end{bmatrix}$$

Example:

$$\mathbf{A} + \mathbf{B} - 3\mathbf{C}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} - 3\mathbf{C} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 10 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1-30 \\ 3+1-3 \\ 6+2-6 \end{bmatrix} = \begin{bmatrix} -27 \\ 1 \\ 2 \end{bmatrix}$$

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Dot Product: Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ be two vectors of length *n*. Then the dot product of the two vectors **u** and **v** is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

A dot product is also an inner product.

Example:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 4 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 7 & 2 \end{bmatrix} = (4 \times 3) + (1 \times 1) + (2 \times 7) + (3 \times 2) = 33$$

Example (divergence of a vector):

$$\nabla \cdot \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} u & v & w \end{bmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Dot Product and Scalar Multiplication Rules:

- $\mathbf{u} \cdot \mathbf{v}$ is a scalar
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot 0 = 0 = 0 \cdot \mathbf{u}$
- $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$
- $(k\mathbf{u}) \cdot \mathbf{v} = (k)\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$ for *k* scalar
- $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

Cross Product: Let $\mathbf{u} = [u_1, u_2, u_3]$ and $\mathbf{v} = [v_1, v_2, v_3]$ be two vectors of length 3. Then the cross product of the two vectors **u** and **v** is defined as

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \hat{i} \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} - \hat{j} \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} + \hat{k} \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

$$= \hat{i}(u_2 v_3 - u_3 v_2) - \hat{j}(u_1 v_3 - u_3 v_1) + \hat{k}(u_1 v_2 - u_2 v_1)$$

Example:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 4 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 7 \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 2 \\ 3 & 1 & 7 \end{bmatrix} = \hat{i}(7-2) - \hat{j}(28-6) + \hat{k}(4-3) = 5\hat{i} - 22\hat{j} + 1\hat{k}$$

Example (curl of a vector):

$$\nabla \times \mathbf{v} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} u & v & w \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \hat{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

<u>**Cross Product Rules**</u>:

- **u** × **v** is a vector
- **u** × **v** is orthogonal to both **u** and **v**
- **u** × 0 = 0 = 0 x **u**
- **u** × **u** = 0
- **u** × **v** = **-**(**v** × **u**)
- (*k***u**) × **v** = *k*(**u** × **v**) **=** **u** × (*k***v**) for any scalar *k*
- **u** × **(v** + **w**) = (**u** × **v**) + (**u** × **w**)
- **(v** + **w**) × **u** = (**v** × **u**) + (**w** × **u**)

NOTE: In general, for a vector **A** and a scalar *k*, *k***A** = **A***k*.

$$k\mathbf{A} = k\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix} = \begin{bmatrix} a_1k \\ a_2k \\ \vdots \\ a_nk \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} k = \mathbf{A}k$$

However, when computing the gradient of a scalar, the scalar product is *not* commutative because ∇ itself is not commutative, i.e.,

$$\nabla T = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} T = \begin{bmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{bmatrix} = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$\neq T\frac{\partial}{\partial x}\hat{i} + T\frac{\partial}{\partial y}\hat{j} + T\frac{\partial}{\partial z}\hat{k}$$

Matrix Algebra:

Matrix Addition: To add two matrices, they both must have the same number of rows and the same number of columns. The elements of the two matrices are simply added together, element by element. Matrix subtraction works in the same way, except the elements are subtracted rather than added.

A + B:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & & & \\ b_{3,1} & & b_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ b_{m,1} & & \cdots & & b_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & a_{1,3}+b_{1,3} & \cdots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & & & \\ a_{3,1}+b_{3,1} & & a_{3,3}+b_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1}+b_{m,1} & & \cdots & & a_{m,n}+b_{m,n} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} + \begin{bmatrix} 100 & 200 & 300 \\ 1000 & 2000 & 3000 \end{bmatrix} = \begin{bmatrix} 101 & 202 & 303 \\ 1010 & 2020 & 3030 \end{bmatrix}$$

Matrix Addition Rules: Let **A**, **B** and **C** denote arbitrary [*m* x *n*] matrices where *m* and *n* are fixed. Let *k* and *p* denote arbitrary real numbers.

- **A** + **B** = **B** + **A**
- **A** + (**B** + **C**) = (**A** + **B**) + **C**
- There is an [*m* x *n*] matrix of 0's such that 0 + **A** = **A** for each **A**
- For each **A** there is an [*m* x *n*] matrix –**A** such that **A** + (-**A**) = 0
- *k*(**A** + **B**) = *k***A** + *k***B**
- (*k*+*p*)**A** = *k***A** + *p***A**
- (*kp*)**A** = *k*(*p***A**)

Matrix Transpose: Let **A** and **B** denote matrices of the same size, and let *k* denote a scalar.

$\mathbf{A}^T$ (also denoted **A'**):

$$\begin{bmatrix} \boldsymbol{a_{1,1}} & \boldsymbol{a_{1,2}} & \boldsymbol{a_{1,3}} & \cdots & \boldsymbol{a_{1,n}} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix}^T = \begin{bmatrix} \boldsymbol{a_{1,1}} & a_{2,1} & a_{3,1} & \cdots & a_{m,1} \\ \boldsymbol{a_{1,2}} & a_{2,2} & & & \\ \boldsymbol{a_{1,3}} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ \boldsymbol{a_{1,n}} & & \cdots & & a_{m,n} \end{bmatrix}$$

$[m \text{ x } n]$ $[n \text{ x } m]$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}^T = \begin{bmatrix} 1 & 10 \\ 2 & 20 \\ 3 & 30 \end{bmatrix}$$

Matrix Transpose Rules:

- If **A** is an $[m \text{ x } n]$ matrix, then $\mathbf{A}^T$ is an $[n \text{ x } m]$ matrix.
- $(\mathbf{A}^T)^T = \mathbf{A}$
- $(k\mathbf{A})^T = k\mathbf{A}^T$
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

Matrix Multiplication: There are several rules for matrix multiplication. The first concerns the multiplication between a matrix and a scalar. Here, each element in the product matrix is simply the element in the matrix multiplied by the scalar.

Scalar Multiplication sA:

$$s\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix} = \begin{bmatrix} s\,a_{1,1} & s\,a_{1,2} & s\,a_{1,3} & \cdots & s\,a_{1,n} \\ s\,a_{2,1} & s\,a_{2,2} & & & \\ s\,a_{3,1} & & s\,a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ s\,a_{m,1} & & \cdots & & s\,a_{m,n} \end{bmatrix}$$

Example:

$$4\begin{pmatrix} 3 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 12 & 24 \end{pmatrix}$$

Matrix Product AB: This is multiplication of a matrix by another matrix. Here, the number of columns in the first matrix must equal the number of rows in the second matrix, e.g., [m × n][n × p] = [m × p].

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & & & \\ b_{3,1} & & b_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ b_{n,1} & & \cdots & & b_{n,p} \end{bmatrix}$$

= [m × p] matrix whose (*i*,*j*) entry is the dot product of the *i*th row of **A** and the *j*th column of **B**.

Example (inner (dot) product):

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = (1\times 4) + (2\times 5) + (3\times 6) = 32$$

[1 x 3][3 x 1] = [1 x 1]

Example (outer product):

$$\begin{bmatrix}4\\5\\6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}=\begin{bmatrix}4\times1 & 4\times2 & 4\times3\\5\times1 & 5\times2 & 5\times3\\6\times1 & 6\times2 & 6\times3\end{bmatrix}=\begin{bmatrix}4 & 8 & 12\\5 & 10 & 15\\6 & 12 & 18\end{bmatrix}$$

[3 x 1][1 x 3] = [3 x 3]

Example (general matrix product):

$$\begin{bmatrix}1 & 2 & 3\\10 & 20 & 30\end{bmatrix}\begin{bmatrix}1 & 10\\2 & 20\\3 & 30\end{bmatrix}=\begin{bmatrix}(1\times1)+(2\times2)+(3\times3) & (1\times10)+(2\times20)+(3\times30)\\(10\times1)+(20\times2)+(30\times3) & (10\times10)+(20\times20)+(30\times30)\end{bmatrix}=\begin{bmatrix}14 & 140\\140 & 1400\end{bmatrix}$$

[2 x 3] [3 x 2] [2 x 2]

<u>Matrix Multiplication Rules:</u> Assume that k is an arbitrary scalar and that A, B, and C, are matrices of sizes such that the indicated operations can be performed.

- **IA = A**, **BI = B**
- **A**(**BC**) = (**AB**)**C**
- **A**(**B** + **C**) = **AB** + **AC**, **A**(**B** – **C**) = **AB** – **AC**
- (**B** + **C**)**A** = **BA** + **CA**, (**B** – **C**)**A** = **BA** – **CA**
- k(**AB**) = (k**A**)**B** = **A**(k**B**)
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

<u>NOTE</u>**:** In general, matrix multiplication is *<u>not</u>* commutative: **AB ≠ BA**

Matrix Division: There is no simple division operation, per se, for matrices. This is handled more generally by left and right multiplication by a matrix inverse.

Matrix Inverse: The inverse of a matrix is defined by the following:

$\mathbf{AB} = \mathrm{I} = \mathbf{BA}$ if and only if **A** is the ***inverse*** of **B**

We then write: $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = 1 = \mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B}$

***NOTE*:** Consider general matrix expression::

$$\begin{aligned} \mathbf{A\,X} &= \mathbf{B} \\ \mathbf{A}^{-1}\,\mathbf{A\,X} &= \mathbf{A}^{-1}\,\mathbf{B} \\ \mathbf{A}^{-1}\,\mathbf{A\,X} &= \mathbf{A}^{-1}\,\mathbf{B} \\ 1\,\mathbf{X} &= \mathbf{A}^{-1}\,\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\,\mathbf{B} \end{aligned}$$

Also note, not all matrices are invertible; e.g., the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Special Matrix Algebra Rules in Matlab:

Matrix + Scalar Addition: A + *s*:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix} + s = \begin{bmatrix} a_{1,1}+s & a_{1,2}+s & a_{1,3}+s & \cdots & a_{1,n}+s \\ a_{2,1}+s & a_{2,2}+s & & & \\ a_{3,1}+s & & a_{3,3}+s & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1}+s & & \cdots & & a_{m,n}+s \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} + 100 = \begin{bmatrix} 101 & 102 & 103 \\ 110 & 120 & 130 \end{bmatrix}$$

Matrix times matrix 'dot' multiplication, A .* B (similar for 'dot' division A ./ B):

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & & a_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1} & & \cdots & & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & & & \\ b_{3,1} & & b_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ b_{n,1} & & \cdots & & b_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & \cdots & a_{1,n}b_{1,m} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & & & \\ a_{3,1}b_{3,1} & & a_{3,3}b_{3,3} & & \vdots \\ \vdots & & & \ddots & \\ a_{m,1}b_{n,1} & & \cdots & & a_{m,n}b_{n,m} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} .* \begin{bmatrix} 10 & 20 & 30 \\ 100 & 200 & 300 \end{bmatrix} = \begin{bmatrix} 10 & 40 & 90 \\ 1000 & 4000 & 9000 \end{bmatrix}$$