Road to Turbulence in Stratified Flow

(Example: Boundary flow with $f = 0$)

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b \delta_{i3} \\
b &= -g \frac{\Delta \rho}{\rho_0} \\
N^2 &= -\frac{g \partial \rho_\theta}{\rho_0 \partial z}
\end{align*}
\]

Turbulence occurs when:

\[
Re = \frac{UL}{\nu} = \text{Reynolds Number} \gg 1
\]

\[
R_i = \frac{V}{\nu} = \text{Richardson Number} < \frac{1}{4}
\]
Advection $u_j \frac{\partial u_i}{\partial x_j}$ and the Cascade Process

**1D example**

\[
\begin{align*}
    u_1 &= A_1 \sin(k_1 x) \quad k_1 = \frac{2\pi}{\lambda_1} \quad \lambda_1 = 1 \\
    u_2 &= A_2 \sin(k_2 x) \quad k_2 = \frac{2\pi}{\lambda_2} \quad \lambda_2 = 1.5
\end{align*}
\]
Kelvin-Helmholtz Instabilities

\[ R_i = \frac{N^2}{\left( \frac{\partial U}{\partial z} \right)^2} < \frac{1}{4} \]
The Path to Turbulent Fluctuations

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b \delta_{i3}
\]

\[u_i = \bar{u}_i + u_i' \quad i = 1, 2, 3 \quad \text{Let } \bar{u}_1 = U(z), \bar{u}_2 = \bar{u}_3 = 0\]

\[u_1' = u'; \quad u_2' = v'; \quad u_3' = w'\]

\[p = \bar{p} + p' = P + p'\]

\[b = B(z) + b' \quad \text{Note: } b' = -g \frac{\rho'}{\rho_0}\]

where

\[B = -g \frac{\rho - \rho_0}{\rho_0}\]
Average Equations

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial z}
\]

\[
0 = -\frac{1}{\rho_0} \frac{\partial P}{\partial z} + B
\]

Equations for turbulent fluctuations

I \[ \frac{\partial u_i'}{\partial t} + U \frac{\partial u_i'}{\partial x} + u_j' \frac{\partial u_i'}{\partial x_j} + w' \frac{\partial u_i'}{\partial z} = -\frac{1}{\rho_0} \nabla p' + \nu \nabla^2 u_i' \quad (i=1,2 \text{ Horizontal}) \]

II \[ \frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} + w' \frac{\partial w'}{\partial z} + w' \frac{\partial U}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \nu \nabla^2 w' + b' \quad \text{Vertical} \]

Multiply (Dot Product) I by \( \cdot u_i' \), I I by \( w' \), Add and Average.

Use: \( \nabla \cdot \bar{u} + \frac{\partial w}{\partial z} = 0 \)
TKE (Turbulent Kinetic Energy) Equation

\[
\frac{D}{Dt}(TKE) = P - B - \varepsilon + Tr
\]

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}
\]

\[
TKE = \text{Turbulent Kinetic Energy} = \frac{1}{2} (\langle (u')^2 + (v')^2 + (w')^2 \rangle)
\]

\[
P = \text{Production Turbulent Energy by Mean Shear} = -\langle u'w' \rangle \frac{\partial \bar{u}}{\partial z}
\]

\[
B = \frac{g}{\rho_0} \langle w' \rho' \rangle = \text{Sink or Source of Turbulent Potential Energy}
\]

\[
\varepsilon = 2\nu \langle s_{ij} s_{ij} \rangle = \text{Dissipation of TKE}
\]

\[
s_{ij} = \frac{1}{2} (\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}) = \text{Turbulent Strain Rate Tensor}
\]

\[
Tr = \text{Transport of TE} = -\frac{\partial}{\partial x_i} \left[ \frac{1}{\rho_0} \langle u'_i p' \rangle + \frac{1}{2} \langle u'_i u'_j u'_j \rangle - 2\nu \langle u'_i s_{ij} \rangle \right]
\]
Steady State Homogeneous TKE Equation

\[ \frac{D}{Dt} (TKE) = 0 \quad Tr = 0 \]

\[ P = B + \varepsilon \]

\[ P = - \langle u'w' \rangle \frac{dU}{dz} \]

\[ B = \frac{g}{\rho_o} \langle \rho'w' \rangle \]
Turbulent Temperature Fluctuations

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \kappa_T \nabla^2 T \quad k_T = \text{thermal diffusivity} = 1.4 \cdot 10^{-7} \frac{m^2}{sec}
\]

Typical Ocean Case \( u_i = U_i + u_i' \) \( U_i = [U(z), 0, 0] \) \( T = \bar{T}(z) + T' \Rightarrow \)

\[
\frac{\partial T'}{\partial t} + U \frac{\partial T'}{\partial x} + w' \frac{\partial \bar{T}}{\partial z} + u' \frac{\partial T'}{\partial z} + v' \frac{\partial T'}{\partial y} + w' \frac{\partial T'}{\partial z} = \kappa_T \nabla^2 T'
\]

Multiply Above Equation by \( T' \) and Average. Assume Stationarity and Homogeneity in horizontal directions

TV(Temperature Variance) Equation

\[
P_T = \frac{\chi}{2}
\]

\[
P_T = - < w' T' > \frac{d \bar{T}}{dz} \quad \chi = 2 k_T < \nabla T' \cdot \nabla T '>
\]

\( P = \) Production of Temperature Variance

\[
\frac{\chi}{2} = \text{Molecular Diffusion of Temperature (heat)}
\]
Steady State Homogenous TKE and TV Equations : Linkage

\[ P = B + \varepsilon \]

\[ P_T = \frac{\chi}{2} \]

\[ P = - < u'w' > \frac{dU}{dz} \]

\[ B = \frac{g}{\rho_o} < \rho'w' > = - g \alpha < w'T' > \]

\[ P_T = - < w'T' > \frac{dT}{dz} \]

\[ \chi = 2k_T < \nabla T' \cdot \nabla T' > \]
Relating $\chi$ to $\varepsilon$

Case of Density only a Function of Temperature, $\rho = \rho(T)$

$$B = \frac{g}{\rho_o} < \rho 'w' > = -g \alpha < w 'T' > = \frac{g \alpha \chi}{2 \frac{dT}{dz}} = \frac{N^2 \chi}{2 (\frac{dT}{dz})^2}$$

$\alpha = \text{thermal expansion coefficient} = 1.4 \cdot 10^{-4} \ (^\circ C)^{-1}$

$$= \frac{1}{\rho_o} \frac{\partial \rho}{\partial T}$$

Define $\gamma = \frac{B}{\varepsilon}$ Mixing Efficiency

$$\Rightarrow \varepsilon = \frac{B}{\gamma} = \frac{\alpha g \chi}{2 \gamma \frac{dT}{dz}} = \frac{N^2 \chi}{2 \gamma (\frac{dT}{dz})^2}$$

Observations show in the ocean that $\gamma \approx 0.2$
**Stratified Turbulence**

Production ($P$) from Mean Shear Flow

Dissipation ($\varepsilon$) to heat

**B) Buoyancy Flux to Potential Energy Mixing**

$R_i = \frac{N^2}{\left(\frac{\partial U}{\partial z}\right)^2} < \frac{1}{4}$

$N^2 = -\frac{g}{\rho_0} \frac{\partial \rho_\theta}{\partial z}$

$\varepsilon$ is rate of energy transfer (Watts/kg) from larger scale to the smallest scales
Characteristic Scales of Turbulence

Key Factor: Cascade of kinetic energy from large to small scales, $\varepsilon$

$$\varepsilon = \frac{\Gamma}{\rho_0 V} \left( \frac{Watts}{kg} \right)$$

Note $\frac{Watts}{kg} = \frac{m^2}{sec^3}$, $\Gamma$ power transferred to smaller scales

Energy Containing (Large) Scales of the Turbulent Field

Scale with $\varepsilon$, $L$

$$u, L, T$$

$$\varepsilon \sim \frac{u^3}{L} \text{ or } L \sim \frac{u^3}{\varepsilon}$$

$$T \sim \frac{L}{u}$$

Dissipative (Small) Scales of the Turbulent Field

Scale with $\varepsilon$, $\nu$

$$v, \eta, \tau$$

$$\eta \sim \left( \frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}}$$

$$v \sim (\varepsilon \nu)^{\frac{1}{4}}$$

$$\tau \sim \left( \frac{\nu}{\varepsilon} \right)^{\frac{1}{2}}$$
Ratio of Smallest (Dissipative) to Largest (Energy Containing) Turbulence Scales

\[ \frac{\eta}{L} \sim \left[ \frac{uL}{v} \right]^{-\frac{3}{4}} = R^{-\frac{3}{4}} \]

\[ \frac{\tau}{T} \sim \left[ \frac{uL}{v} \right]^{-\frac{1}{2}} = R^{-\frac{1}{2}} \Rightarrow \frac{\Omega}{\omega} \ll 1 \]

\[ \frac{v}{u} \sim \left[ \frac{uL}{v} \right]^{-\frac{1}{4}} = R^{-\frac{1}{4}} \]

\[ R = \text{Turbulent Reynolds number} \]

Typical oceanic values

\[ R = \frac{uL}{v} = \frac{\{10^{-2},10^{-3}\}}{10^{-6}} = 10^4,10^3 \]

\[ \omega = \left[ \frac{\varepsilon}{v} \right]^{\frac{1}{2}} \sim .1,1 \text{ sec}^{-1} \text{ while } \Omega = \frac{\partial U}{\partial z} \sim .01,.001 \text{ sec}^{-1} \]

\[ 10^{-6} \frac{\text{watts}}{\text{kg}} < \varepsilon < 10^{-9} \frac{\text{watts}}{\text{kg}} \]

\[ L = 1m \]

\[ u = (\varepsilon l)^{\frac{1}{3}} \sim 10^{-2},10^{-3} \frac{m}{\text{sec}} \]
Steady State, Homogenous, No Stratification, B=0, \( \rightarrow P = \varepsilon \)

\[
P = - < u' w' > \frac{\partial U}{\partial z} = \varepsilon = v < \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} > = 2v < s_{ij} s_{ij} > = \text{Dissipation of TKE}
\]

Turbulent Strain Rate \( s_{ij} = \frac{1}{2} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right) \Rightarrow s_{ij} \sim \left( \frac{\varepsilon}{v} \right) \frac{1}{2} = \left( \frac{u^3}{Lv} \right) \frac{1}{2} \)

Mean Strain Rate \( S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \Rightarrow S_{ij} \sim \left( \frac{u}{L} \right) \)

\[
\frac{s_{ij}}{S_{ij}} \sim R^2 \quad \text{Note for } U = U(z), \quad V = W = 0 \quad S_{13} = \frac{1}{2} \frac{\partial U}{\partial z} \quad \left( \frac{u}{L} \right)
\]

\[
\overline{s_{ij} s_{ij}} \gg S_{ij} S_{ij} \quad (3.2.8)
\]

The fluctuating strain rate \( s_{ij} \) is thus very much larger than the mean rate of strain \( S_{ij} \) when the Reynolds number is large. Since strain rates have the dimension of \( \text{sec}^{-1} \), this implies that the eddies contributing most to the dissipation of energy have very small convective time scales compared to the time scale of the flow. This suggests that there should be very little direct interaction between the strain-rate fluctuations and the mean flow if the Reynolds number is large. In other words, \( S_{ij} \) and \( s_{ij} \) do not interact strongly, because they are not tuned to the same frequency band. Therefore, the small-scale structure of turbulence tends to be independent of any orientation effects introduced by the mean shear, so that all averages relating to the small eddies do not change under rotations or reflections of the coordinate system. If this is the case, the small-scale structure is called isotropic (Figure 3.2). Isotropy at small scales is called local isotropy (see Chapter 8).
Isotropy at Dissipation Scales

\[ \varepsilon = \nu \left| \frac{\partial u'_i}{\partial x_j} \right| \frac{\partial u'_i}{\partial x_j} \]

where

\[ < \left( \frac{\partial u'_1}{\partial x_1} \right)^2 > = < \left( \frac{\partial u'_2}{\partial x_2} \right)^2 > = < \left( \frac{\partial u'_3}{\partial x_3} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_1}{\partial x_2} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_1}{\partial x_3} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_2}{\partial x_1} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_2}{\partial x_3} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_3}{\partial x_1} \right)^2 > = \frac{1}{2} < \left( \frac{\partial u'_3}{\partial x_2} \right)^2 > \]

Only one component of turbulent shear is required to obtain \( \varepsilon \)

\[ \varepsilon = 15 \nu \left( \frac{\partial u'_1}{\partial x_1} \right)^2 = \left( \frac{\partial u'_2}{\partial x_2} \right)^2 = \left( \frac{\partial u'_3}{\partial x_3} \right)^2 \]

\[ = \frac{15}{2} \nu \left( \frac{\partial u'_1}{\partial x_2} \right)^2 = \frac{15}{2} \nu \left( \frac{\partial u'_1}{\partial x_3} \right)^2 = \frac{15}{2} \nu \left( \frac{\partial u'_2}{\partial x_1} \right)^2 = \frac{15}{2} \nu \left( \frac{\partial u'_2}{\partial x_3} \right)^2 = \frac{15}{2} \nu \left( \frac{\partial u'_3}{\partial x_1} \right)^2 = \frac{15}{2} \nu \left( \frac{\partial u'_3}{\partial x_2} \right)^2 \]
Taylor Microscale $\lambda$

$$\lambda^{-2} = \frac{1}{2} < (u')^2 > < \left( \frac{\partial u'}{\partial x_2} \right)^2 >$$

$$= \frac{1}{< (u')^2 >} < \left( \frac{\partial u'}{\partial x_1} \right)^2 > \Rightarrow$$

$$\varepsilon = \frac{15}{2} \nu < \left( \frac{\partial u'}{\partial x_2} \right)^2 > = 15\nu \frac{< (u')^2 >}{\lambda^2}$$

$$\frac{u^3}{L} \approx \frac{15\nu u^2}{\lambda^2} \Rightarrow \left( \frac{\lambda}{L} \right)^2 = \frac{15\nu}{uL} = \frac{15}{R}$$

$$\frac{\lambda}{L} = \sqrt{\frac{15}{R}}$$

using $\frac{\eta}{L} = R^{-\frac{3}{4}}$  \hspace{1cm} $\frac{\lambda}{\eta} = \sqrt{15} R^{\frac{1}{4}}$

Note: $L >> \lambda >> \eta$ when $R >> 1$
Example: Wind Tunnel Decay of Turbulence

\[ \frac{d}{dt} \left( TKE \right) = -\varepsilon \]

\[ \frac{d}{dt} \left( \frac{3}{2} u^2 \right) = -\frac{u^3}{l} \]

Use \( l = \sqrt{2kt + L^2} \approx \sqrt{2kt} \) \( k = u_0 L \) for a time \( t \) such that \( l \gg L \)

\[ 3 \frac{du}{dt} = -\frac{u^2}{\sqrt{2kt}} \Rightarrow \]

\[ u = \frac{u_0}{1 + \left( \frac{t}{\tau} \right)^{\frac{1}{2}}} \]

where \( \tau = \frac{9k}{2(u_0)^2} \)
$$0 = P - \text{Tr} - \varepsilon$$

$$0 = -u_1 u_2 \frac{\partial U_1}{\partial x_2} - \frac{\partial}{\partial x_2} \left( \frac{1}{\rho} u_2 \rho + \frac{1}{2} u_i u_j u_2 \right) - \varepsilon. \quad (3.2.38)$$

TKE $(x, z, y)$ Component Equations

$$0 = -u_1 u_2 \frac{\partial U_1}{\partial x_2} + \frac{1}{\rho} p \frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2} \left( \frac{1}{2} u_1^2 u_2 \right) - \frac{1}{3} \varepsilon, \quad (3.2.39)$$

$$0 = 0 + \frac{1}{\rho} p \frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_2} \left( \frac{p}{\rho} + \frac{1}{2} u_2^2 \right) u_2 - \frac{1}{3} \varepsilon, \quad (3.2.40)$$

$$0 = 0 + \frac{1}{\rho} p \frac{\partial u_3}{\partial x_3} + \frac{\partial}{\partial x_2} \left( \frac{1}{2} u_3^2 u_2 \right) - \frac{1}{3} \varepsilon. \quad (3.2.41)$$

$$\rho \frac{\partial u_1}{\hat{x}_1} + p \frac{\partial u_2}{\hat{x}_2} + p \frac{\partial u_3}{\hat{x}_3} + p \frac{\partial u_i}{\hat{x}_i} = 0. \quad (3.2.42)$$

From T/L page 73

Red: Dissipation (Isotropic)

Brown: Redistribution by Pressure fluctuations
Role of Vorticity in Turbulent Cascade

\( \nabla \times \{ \text{Navier Stokes Equation} \} = \text{Vorticity Equation} \)

\[
\nabla \times \left\{ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right\} = \nabla \times \left\{ -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \right\}
\]

\[
\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega}
\]

Inviscid Case (no Friction)]

\[
\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j}
\]

\[
\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
\]

\[
= s_{ij} + r_{ij}
\]

Using \( r_{ij} = -\varepsilon_{ijk} \omega_k \)

\[
\frac{d \omega_i}{dt} = \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j s_{ij}
\]
Vorticity Equation
\[
\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \nabla \vec{\omega} = \vec{\omega} \nabla \vec{u} + \nu \nabla^2 \vec{\omega}
\]
Inviscid Case (no Friction)]
\[
\frac{d \omega_i}{dt} = \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j s_{ij}
\]
\[
S_{ij} = \begin{bmatrix}
\frac{\partial U}{\partial x} & 0 & 0 \\
0 & \frac{\partial V}{\partial y} & 0 \\
0 & 0 & \frac{\partial W}{\partial z}
\end{bmatrix}
\]
\[
S_{22} = S_{33} = -\frac{1}{2} S_{11}
\]
\
Wind Tunnel contraction
\[
\frac{d \omega_1}{dt} = \omega_1 s_{11} \Rightarrow \omega_1 = \omega_0 \exp(s_{11} t)
\]
\[
\frac{d \omega_2}{dt} = \omega_2 s_{22} \Rightarrow \omega_2 = \omega_0 \exp(-\frac{1}{2} s_{11} t)
\]
\[
\frac{d \omega_3}{dt} = \omega_3 s_{33} \Rightarrow \omega_3 = \omega_0 \exp(-\frac{1}{2} s_{11} t)
\]
Steady State Vorticity Variance Equation

\[
\frac{d}{dt} \langle \omega_i'\omega_i' \rangle + (\ldots) + (\ldots) + \ldots + \langle \omega_i'\omega_j's_{ij} \rangle = \nu \frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_i'}{\partial x_j}
\]

Order of Magnitude scaling results in (Pages 87-91 T/L)

\[
\langle \omega_i'\omega_j's_{ij} \rangle = \nu \frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_i'}{\partial x_j}
\]

Key Points

1. Unlike TKE Equation no Source (production) term.
2. Right Hand Side (RHS) represents decay and is always positive; thus LHS is positive.
3. LHS arises represents vortex stretching by strain rate. RHS > 0 implies more vortex stretching than squeezing.
4. Vorticity is then transferred to smaller scales!
Cascade Mechanism Role of Vorticity Stretching

\[ s'_{ij} = \begin{bmatrix} \frac{\partial u'}{\partial x} & 0 & 0 \\ 0 & \frac{\partial v'}{\partial y} & 0 \\ 0 & 0 & \frac{\partial w'}{\partial z} \end{bmatrix} = \begin{bmatrix} s'_{11} & 0 & 0 \\ 0 & s'_{22} & 0 \\ 0 & 0 & s'_{33} \end{bmatrix} \]

Let \( s'_{22} = s'_{33} = -\frac{1}{2}s'_{11} \) \& \( \omega_2' \approx \omega_3' \)

\[
< \omega_i' \omega_j's'_{ij} > = \nu < \frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_j'}{\partial x_j} > 
\]

\[
\{< (\omega_1')^2 > - < (\omega_2')^2 > \} \times \{s'_{11}\} > 0 \quad \Rightarrow 
\]

if \( s'_{11} > 0 \), \( \{< (\omega_1')^2 >\} > \{< (\omega_2')^2 >\} \)

if \( s'_{11} < 0 \), \( \{< (\omega_2')^2 >\} > \{< (\omega_1')^2 >\} \)

- Both stretching and squeezing results in an imbalance in vorticity variance
- Imbalances are dissipated
- Vortices in direction of larger variances are reduced in size (wind tunnel example)
Convective Boundary Layer (CBL)

- $H$ = depth of CBL

- $L$ = Monin-Oboukhov (MO) depth

- $z' < L$  Dominated by Wind Stress Turbulence, $u^*$

- $z' > L$  Dominated by Buoyancy Driven Turbulence, $w^*$

- $H$ = depth of CBL from balancing rotation and buoyancy
Convective Boundary Layer (CBL)

Wind

Surface (s)

\[ P = B + \varepsilon \quad B < 0 \]

\[ P = - \langle u' w' \rangle \frac{dU}{dz} \]

for a log layer \( U = u^* \log \left( \frac{z'}{z_0} \right) \)

\[ P = (u^*)^2 \frac{u^*}{.4 z} \]

\[ B = \frac{g}{\rho_0} \langle \rho' w' \rangle \]

\[ L = - \frac{(u^*)^3}{.4 z} \]

\[ B = - \frac{(u^*)^3}{.4 L} \]

\[ R_f = \frac{B}{P} = \frac{\rho_0}{(u^*)^3} \]

\[ L = \text{Monin Obhukov (MO) depth} \]

\[ h = \text{Depth of CBL} \]

\[ h = \sqrt{\frac{2k}{f}} \quad \text{using} \quad k = u^* h \Rightarrow h \approx \frac{2u^*}{f} \]
In winter time the wind blows steadily over a region of the continental shelf at \( U_{10} = 10 \frac{m}{sec} \) and loses heat at a rate of \( H = 400 \frac{watts}{m^2} \). The depth of water is found to be \( D = 400m \); Take \( f = 10^{-4} \text{ sec}^{-1} \). (a) How deep do you estimate the convective boundary layer to be? (b) Estimate the turbulent velocity at \( z = 0.50m, 200, 350m \)? (c) If the wind \( U_{10} \) decreases by 50% how does this change these answers?

\[
(a) \quad \frac{\tau}{\rho_w} = (u^*)^2 = \frac{\rho_a C_D (U_{10})^2}{\rho_w} = \frac{(1 \frac{kg}{m^3})2.5 \cdot 10^{-3} (10 \frac{m}{sec})^2}{1000 \frac{kg}{m^3}} \Rightarrow u^* = 0.016 \frac{m}{sec}
\]

\[
h = \frac{2 \cdot u^*}{f} = \frac{2 \cdot 0.016 \frac{m}{sec}}{10^{-4} \text{ sec}^{-1}} = 320m
\]
Wind

**Example Problem (Continued)**

\[ L = - \frac{(u^*)^3}{\frac{g}{\rho_v} \cdot 4 <\rho' w'>} = - \frac{(u^*)^3}{g \alpha .4 <T'w'>} = - \frac{(u^*)^3}{g \alpha .4 \frac{H}{\rho c_p}} \]

\[ = \frac{(.016 \text{ m})^3}{\text{sec}} = 78 \text{ m} \]

\[ 9.8 \frac{m}{\text{sec}^2} \cdot 1.4 \cdot 10^{-4} \left(\circ C\right)^{-1.4} \cdot \frac{400 \text{ watts}}{m^2} \cdot \frac{1000 \text{ kg}}{m^3} \cdot 4.2 \cdot 10^3 \frac{J}{\text{kg} \cdot \left(\circ C\right)} \]

\[ z = 0.50 \text{ m} , \ z < L = 78 \text{ m} \quad \Rightarrow \quad u_{\text{turb}} = u^* = .016 \frac{m}{\text{sec}} \]

\[ z = 200 > 78 \quad \Rightarrow \quad z = 200 < h = 320 \text{ m} \quad \Rightarrow \quad w^* \approx u^* \left( \frac{200}{78} \right)^{\frac{1}{3}} = .021 \frac{m}{\text{sec}} \]

\[ z = 350 > h \quad \Rightarrow \quad w^* \approx u^* = 0 \]
(b) \( U_{10} = 5 \frac{m}{\text{sec}} \)

\[
\frac{\tau}{\rho_w} = (u*)^2 = \frac{\rho_w C_D (U_{10})^2}{1000 \frac{kg}{m^3}} = \frac{(1 \frac{kg}{m^3})2.5 \cdot 10^{-3} (5 \frac{m}{\text{sec}})^2}{1000 \frac{kg}{m^3}} \Rightarrow u* = .008 \frac{m}{\text{sec}}
\]

\[
h = \frac{2u*}{f} = \frac{2 \cdot .008}{10^{-4}} \frac{m}{\text{sec}^{-1}} = 160m
\]

\[
L = - \frac{(u*)^3}{\frac{g}{\rho_0} \cdot 4 \rho w} = \frac{(u*)^3}{g \alpha 4 \theta w} = \frac{(u*)^3}{g \alpha 4 \frac{H}{\rho c_p}}
\]

\[
= \frac{(.008 \frac{m}{\text{sec}})^3}{9.8 \frac{m}{\text{sec}^2} \cdot 1.4 \cdot 10^{-4} (^\circ \text{C})^{-1} \cdot 4} = 19.5m
\]
\[ z = 0 \text{, } z < L = 19.5m \Rightarrow u_{turb} = u^* = 0.008 \frac{m}{\text{sec}} \]

\[ z = 50 > L \text{&} < h = 160m \quad w^* \approx u^* \left( \frac{50}{19.5} \right)^{\frac{1}{3}} = 0.011 \frac{m}{\text{sec}} \]

\[ z = 200, 350 > h \Rightarrow w^* \approx u^* = 0 \]
Concept of Buoyancy frequency \( N \)

Block of fluid oscillates at \( N \)

\[
N^2 = -\frac{g}{\rho} \left( \frac{\partial \rho}{\partial z} \right) = -\frac{g}{\rho} \left\{ \frac{\partial \rho}{\partial z} - \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial z} \right\}
\]

Subtract out effect of pressure changes

\[
\frac{\partial \rho}{\partial P} = \frac{1}{c^2} \quad c \text{ the speed of sound } c = 1500 \frac{m}{\text{sec}}
\]

\[
\frac{\partial P}{\partial z} = -\rho g
\]
\[ N^2 = -\frac{g}{\rho} \frac{\partial \rho}{\partial z} - \frac{g^2}{c^2} \]

\[ \frac{g^2}{c^2} \approx 4.4 \cdot 10^{-5} \text{ sec}^{-2} \]

If \( \rho = \rho(T) \) \( \Rightarrow \) \(-\frac{g}{\rho} \frac{\partial \rho}{\partial z} = -\frac{g}{\rho} \frac{\partial \rho}{\partial T} \frac{\partial T}{\partial z} = \alpha \frac{\partial T}{\partial z} \)

\[ \alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} = 1.4 \times 10^{-4} (^{\circ} C)^{-1} \]

\[ N^2 = g \alpha \frac{\partial T}{\partial z} - \frac{g^2}{c^2} \]

If \( \frac{\partial T}{\partial z} \gg \frac{g}{\alpha c^2} = .03 \frac{^{\circ} C}{m} \) \( \Rightarrow \)

\[ N^2 = g \alpha \frac{\partial T}{\partial z} \]