Lagrangian & Eulerian Descriptions

Fluid parcel moves

If we follow a fluid parcel we call it Lagrangian. If we stay at a fixed point \( x \) and time \( t \) we call it Eulerian.

\[ \frac{\dot{x}}{t} \]

\[ U(x, t) \]

\[ \phi (x, t) \]

We can write for a fluid element path

\[ \dot{x} = \dot{x}(x, t, t_0) \]

Let's take to = 0

\[ \dot{x} = x(x, t) \]

For example temperature \( T \)

\[ T = T(x, t) \quad \text{Eulerian} \]

\[ T = T[x(x, t), t] = T[x(x, t)] \quad \text{Lagrangian} \]
Note by definition
\[ x(\vec{x}, t) \big|_{t=t_0} = \vec{x} \]

Let us examine the relationship of Eulerian velocity \( \vec{u}(\vec{x}, t) \) to Lagrangian fluid element path \( \dot{x}(\vec{x}, t) \).

When \( \dot{x}_i \to 0 \)

\[ \vec{u} = \frac{\dot{x}_2 - \dot{x}_1}{t_2 - t_1} \]

Then the Eulerian velocity \( \vec{u}(\vec{x}, t) \) is the Lagrangian displacement path first derivative.
Consider some quantity $\phi$ along a Lagrangian path from $x_0$ to $x$, $t_0$ to $t$.

What is \( \frac{\partial \phi}{\partial t} \) time derivative along Lagrangian path but at a fixed point $x(t)$ and time $t$ we can write:

\[
\phi = \phi(x(t), t)
\]

\[
\frac{\partial \phi}{\partial t} = \left( \frac{\partial \phi}{\partial t} \right)_t + \left( \frac{\partial \phi}{\partial x} \right)_t \frac{\partial x}{\partial t}
\]

\[
= \left( \frac{\partial \phi}{\partial t} \right)_t + \nabla \cdot \nabla \phi
\]

Advection term

Local time derivative
What if
\[ \frac{d\phi}{dt} = 0 \]

\( \phi \) is constant along a particle path. Conserved scalar,

\[ u = 0 \]

Suppose \( u \) is constant,

\[ \frac{d\phi}{dt} = -\frac{\partial \phi}{\partial x} + u \frac{\partial \phi}{\partial x} \]

\[ \frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x} \quad \Rightarrow \quad \phi = \phi(x - ut) \]

At \( t = 0 \) \( \phi = \phi(x) \)
At \( t \) \( x = 0 \) \( \phi = \phi(0) \)

Time change is entirely from advection. The variability is "frozen fixed", also called
taylor hypothesis. (\( u \) constant)
Example of Lagrangian Displacement

\( u = \alpha z \) (Shear Flow)

\[
\begin{align*}
\frac{d\xi}{dt} &= \alpha z = x = \alpha z t + x_0 \\
\frac{d\eta}{dt} &= 0 \\
\frac{d\zeta}{dt} &= \beta z = \beta \zeta
\end{align*}
\]

\[Z_1, \quad u = \alpha z_1, \quad B \quad -\quad C \quad B' \quad C'
\]

\[Z = 0, \quad u = 0 \quad A \quad \xi = \theta \quad \eta = \theta \]

\[A = (0,0) \quad B = (0, z_1) \quad A' = A \quad B' = (\xi_2, z_1 \zeta_2) \]

\[B = (-\xi_1, 0) \quad C = (-\xi_1, \zeta_1) \quad D' = D \quad C' = (\xi_1, \zeta_1, z_1) \]
\[
\tan \theta = \frac{\frac{2L}{\alpha}}{\alpha t} = \frac{1}{\alpha t}
\]

\[t = 0 \quad \tan \theta = \infty \quad \theta = \frac{\pi}{2}\]

\[t = \infty \quad \tan \theta = 0 \quad \theta = 0\]

\[
\phi = \frac{\pi}{2} - \theta
\]

\[
\tan \phi = \alpha \cdot \dot{t}
\]

\[
\frac{d (\tan \phi)}{dt} = \alpha
\]

\(\alpha\) related to a rate of rotation
Strain Rate Tensor

Concept of strain

\[ \varepsilon = \frac{\Delta l}{l} = \frac{\delta x}{\delta x} \]

Strain Rate

\[ \frac{d}{dt} = \frac{\partial}{\partial t} \frac{\delta x}{\delta x} \]

\[ = \frac{\partial u}{\partial x} \]

\[ = \frac{\partial u}{\partial x} \]

\[ = \frac{\partial u}{\partial x} \]

\[ \frac{\partial u}{\partial x} \text{ is not summed} \]

\[ \text{is normal strain rate} \]

\[ \ell = 1, 2, 3 \]
\[
\frac{\partial u_i}{\partial u_j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial \xi_j} - \frac{\partial u_j}{\partial \xi_i} \right)
\]

\[
= s_{ij} + \frac{1}{2} r_{ij}
\]

Note \( s_{ij} = s_{ji} \) Strain rate tensor

\( r_{ij} = - r_{ij} \) Rotation tensor

Note \( s_{ii} = 0 \). It is always possible to rotate coordinates such that in the new coordinate system \( s'_{ij} = 0 \) \ For \( i \neq j \) (only diagonal elements) \[
\begin{bmatrix}
    s'_{11} & 0 & 0 \\
    0 & s'_{22} & 0 \\
    0 & 0 & s'_{33}
\end{bmatrix}
\]
Conservation of Mass and Volume

\[ \delta V = \delta V \Rightarrow \delta x \delta y \delta z = \delta X \delta Y \delta Z \]

Let \( \tilde{v} = \frac{\delta v}{\delta V} \)

\[ \left( \frac{\partial \tilde{v}}{\partial t} \right)_v = 0 \Rightarrow \frac{\partial x}{\partial t} \frac{\delta y \delta z}{\delta X \delta Y \delta Z} + \frac{\partial y}{\partial t} \frac{\delta x \delta z}{\delta X \delta Y \delta Z} + \frac{\partial z}{\partial t} \frac{\delta y \delta z}{\delta X \delta Y \delta Z} = 0 \]

\[ \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \bar{u} = 0 \] (Called Incompressible Condition)
More generally consider the mass in of a fluid element \( \nu \).

Then if mass is conserved

\[
\nu = \nu \quad \text{initial time}
\]

\[
\frac{d\nu}{dt} = 0
\]

\[
\frac{d}{dt} \left( \rho \frac{\partial \nu}{\partial x} \right) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \rho \frac{\partial \omega}{\partial x} = 0
\]

\[
\frac{\partial \omega}{\partial t} + \frac{2}{3} \left( \frac{\partial u}{\partial x} \right)^2 = 0
\]
This is more general and is true for all fluids. (Starting point in acoustics)

If \( \nabla \cdot \mathbf{u} = 0 \) \[ \frac{df}{dt} = 0 \]

Incompressible fluids have \( \frac{df}{dt} = 0 \)

In Ocean density does change.

[Diagram of depth and density]

Explain this!
\[ \frac{dP}{dt} = \frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla P \]

local
advective
change
change

In ocean: \[ \frac{dP}{dt} = 0 \]

\[ \frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla P \]

but be
careful