Turbulent Mean Flow Effects: Inclusion of Rotation

**Horizontal Equation of Motion**

\[
\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -\frac{1}{\rho} \nabla_h p + \frac{1}{\rho} \frac{\partial}{\partial z} \tau
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla
\]

\[
\tau = -\frac{\langle \vec{u}' \vec{w}' \rangle}{\rho} = k_z \frac{\partial \vec{u}}{\partial z}
\]

\[
\frac{\partial u_i}{\partial t} + u \frac{\partial u_i}{\partial x} + v \frac{\partial u_i}{\partial y} + w \frac{\partial u_i}{\partial z} + (\vec{f} \times \vec{u})_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + k_z \frac{\partial^2 u_i}{\partial z^2} + A_h \left( \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} \right)
\]

**Added horizontal Friction (eddies)**
Road to Geostrophy: Dimensional Analysis

<table>
<thead>
<tr>
<th>Acceleration</th>
<th>Advection</th>
<th>Coriolis</th>
<th>Pressure Gradient</th>
<th>Vertical Friction</th>
<th>Horizontal Friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial u_i}{\partial t} )</td>
<td>( u \frac{\partial u_i}{\partial x} + v \frac{\partial u_i}{\partial y} + w \frac{\partial u_i}{\partial z} )</td>
<td>( \overrightarrow{f} \times \overrightarrow{u} ) ( i )</td>
<td>( -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} )</td>
<td>( k_z \frac{\partial^2 u_i}{\partial z^2} )</td>
<td>( A_h (\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}) )</td>
</tr>
</tbody>
</table>

\[
\frac{U}{T} = \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{WU}{H} = \frac{U^2}{L}
\]

\( fU \)

Divide by fU

<table>
<thead>
<tr>
<th>( Ro = \frac{U}{fL} ) Rossby Number</th>
<th>( \frac{k_z U}{H^2} ) Vertical Ekman Number</th>
<th>( \frac{A_h U}{L^2} ) Horizontal Ekman Number</th>
</tr>
</thead>
</table>

If \( Ro << 1, \frac{k_z}{fH^2} << 1, \frac{A_h}{fL^2} << 1 \)

Geostrophy \( \overrightarrow{f} \times \overrightarrow{u} \) \( i \) = \( -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \)
“Large Scale” (Non Turbulent) Oceanic Characteristic Scales

Open Ocean

Coastal Ocean

L ~ 10^6 m    H ~ 10^3 m

f = 10^{-4} \text{ sec}    U ~ 10^{-1} \frac{m}{\text{ sec}}

k = 10^{-1} - 10^{-4} \frac{m^2}{\text{ sec}}    A = 10 - 10^5 \frac{m^2}{\text{ sec}}

Ro = \frac{U}{fL} = 10^{-3}

E_z = \frac{k}{fH^2} = (10^{-3} - 10^{-6})

E_h = \frac{A}{fL^2} = (10^{-3} - 10^{-7})

L ~ 10^4 m    H ~ 10^2 m

f = 10^{-4} \text{ sec}    U ~ 10^{-1} \frac{m}{\text{ sec}}

k = 10^{-1} - 10^{-4} \frac{m^2}{\text{ sec}}    A = 10 - 10^3 \frac{m^2}{\text{ sec}}

Ro = \frac{U}{fL} = 10^{-1}

E_z = \frac{k}{fH^2} = (10^{-1} - 10^{-4})

E_h = \frac{A}{fL^2} = (10^{-1} - 10^{-3})
Geostrophy

Northern Latitude
Geostrophic Equations
\[
\begin{align*}
\frac{\partial p}{\partial x} &= \rho f v \\
\frac{\partial p}{\partial y} &= -\rho f u
\end{align*}
\]

Hydrostatic Equation
\[
\frac{\partial p}{\partial z} = -\rho g
\]

Barotropic Flow \( \rho = \rho_0 \)

\[ x \rightarrow \alpha = \frac{\partial \zeta}{\partial x} > 0 \]

\( z=D \)

\( z=0 \)

\( p_\text{A} > p_\text{B} \)

\[ p = \rho_0 g (D - z + \zeta) \]

\[ \frac{\partial p}{\partial x} = \rho_0 g \alpha \]

\[ v = \frac{1}{\rho_0 f} \frac{\partial p}{\partial x} = \frac{g \alpha}{f} \]
Example: The sea surface of a certain marginal sea of uniform density and depth of 150 m slopes to the north with a change in sea surface height of 1 cm every 10 km. Assume that it is in the northern hemisphere. If the latitude of this sea is such that the Coriolis frequency is given by \( f = 10^{-4} \text{ sec}^{-1} \)

(a) what is the magnitude and direction of the current at the surface, \( z = 0 \)?

(b) at mid depth, \( z = 75 \text{ m} \)?

© Explain in words how your answer to (a) and (b) would change if this sea were located at a latitude lower and latitude higher than assumed above. Assume the same sea surface height and depth for this part.
\[ \alpha = \frac{10^{-2} \, m}{10^4 \, m} = 10^{-6} \]

\[ v = \frac{g \alpha}{f} = \frac{9.8 \, \frac{m}{\text{sec}^2}}{10^{-4}} 10^{-6} \approx 0.1 \, \frac{m}{\text{sec}} \]
Ekman Dynamics

wind

wind stress

Ekman layer

depth of frictional influence, $D$

Coriolis force

average motion of the Ekman layer
Ekman Dynamics

\[
\frac{Du}{Dt} - fV = - \frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right) + \frac{1}{\rho} \frac{\partial \tau_x}{\partial z}
\]

\[
\frac{Dv}{Dt} + fU = - \frac{1}{\rho} \left( \frac{\partial p}{\partial y} \right) + \frac{1}{\rho} \frac{\partial \tau_y}{\partial z}
\]

**Example A:** Steady state, no pressure gradient, wind blows in “y” direction, constant density

\[
fU = \frac{1}{\rho_0} \frac{\partial}{\partial z} \tau_y
\]

\[
q = Y \int_{-D}^{0} u dz = \frac{Y \{\tau_y\}_s}{f \rho_0}
\]

where \( Y \) is length over which the stress occurs.

\( q = \text{Volume Transport} \)

\( M = \rho_0 q = \text{mass or Ekman transport} \)

**Note:** \( D \) is defined by \( \{\tau_y\}_{z=-D} = 0 \)
Ekman Transport

\[
q = \text{Volume Transport} = Y \int u \, dz = \frac{Y}{f} \frac{\{\tau_y\}_s}{\rho_0} = \frac{Y}{f} (u^*)^2
\]

**Note:**

\(X, Y\) are the characteristic lengths over which the wind stress occurs and transport occur.
Coastal Upwelling

\[ \frac{\partial u}{\partial x} = - \frac{\partial w}{\partial z} \Rightarrow w = -\int \frac{\partial u}{\partial x} \, dz \approx \frac{(-q)}{XY} = \frac{(u^*)^2}{Xf} \]
Upwelling Example

Wind \((U_{10})\) blows to the south off of Northern California just north of Monterey Bay (latitude 37°N) at steady rate of 10 m/sec. If the coastal region being affected has dimensions, \(X = 20 \text{ km}, Y = 50 \text{ km}\), as shown in the figure below, estimate:

(a) the Ekman volume transport, \(q\),
(b) upwelling velocity, \(w\), and
(c) how long it would take nutrients located at a depth of 100 m to rise to the surface.

Note:
\[
f = 2 \Omega \sin(\theta) = 2 \cdot \frac{2 \pi \sin(37 \cdot \frac{\pi}{180})}{24 \cdot 3600} \text{ sec}^{-1} = 8.8 \cdot 10^{-5} \text{ sec}^{-1}
\]

\[
(u^*)^2 = \frac{\tau}{\rho_{\text{water}}} = \frac{\rho_{\text{air}} C_D U_{10}^2}{\rho_{\text{water}}} = \frac{1 \cdot \frac{\text{kg}}{\text{m}^3} \cdot 2.5 \cdot 10^{-3} \cdot (10 \text{ m})^2}{1000 \cdot \frac{\text{kg}}{\text{m}^3}} = 2.5 \cdot 10^{-4} \left(\frac{\text{m}}{\text{sec}}\right)^2
\]
Upwelling Example Continued

(a) \[ q = \frac{Y}{f} (u^*)^2 = \frac{5 \cdot 10^4 \text{m}}{8.8 \cdot 10^{-5} \text{sec}^{-1}} \cdot 2.5 \cdot 10^{-4} \left(\frac{m}{\text{sec}}\right)^2 = 1.42 \cdot 10^5 \frac{m^3}{\text{sec}} = .142 \text{Sv} \]

(b) \[ w = \frac{q}{XY} = \frac{(u^*)^2}{Xf} = \frac{2.5 \cdot 10^{-4} \left(\frac{m}{\text{sec}}\right)^2}{2.0 \cdot 10^4 \text{m} \cdot 8.8 \cdot 10^{-5} \text{sec}^{-1}} = 1.4 \cdot 10^{-4} \frac{m}{\text{sec}} = 12 \frac{m}{\text{day}} \]

(c) \[ t = \frac{H}{w} = \frac{100 \text{m}}{12 \frac{m}{\text{day}}} = 8 \text{ days} \]

1 \text{ Sv} = 10^6 \frac{m^3}{\text{sec}}
Ekman Dynamics: Upper Ocean Current Field
(Assumptions: steady state, no surface slope, \( p=0 \), infinitely deep water, constant density \( \rho = \rho_0 \))

**Eddy Viscosity Assumption**

\[-fv = \frac{1}{\rho} \frac{\partial \tau_x}{\partial z} \]

\[fu = \frac{1}{\rho} \frac{\partial \tau_y}{\partial z} \]

\[\tau_x = \rho_0 k \frac{\partial u}{\partial z} \]

\[\tau_y = \rho_0 k \frac{\partial v}{\partial z} \]

**Surface Boundary Condition**

\[|\vec{\tau}_s| = \rho_0 k \frac{\partial u}{\partial z} |_{z=0} = \rho_0 (u^*)^2\]

Specify Direction of \( \vec{\tau}_s \)

\[b = \sqrt{\frac{f}{2k}}\]

Note: \( b^{-1} \) is the "effective" Ekman depth

where \(|\vec{u}| = e^{-1} U = .37 U\)

\(bH >> 1\)

(Deep water assumption)

\[\tilde{z} = -\frac{1}{b}\]

Note: \( \tilde{z} = -z \)

\(\tau = 0\)
Solution for \( \{ \tau_x (\tilde{z}) \}_{\tilde{z}=0} = 0 \) \( \{ \tau_y (\tilde{z}) \}_{\tilde{z}=0} = \tau_s \)

\[
\begin{align*}
u &= U \exp[b \tilde{z}] \cos(b \tilde{z} + \frac{\pi}{4}) \\
v &= U \exp[b \tilde{z}] \sin(b \tilde{z} + \frac{\pi}{4}) \\
U &= \frac{\tau_y}{\rho_0 \sqrt{f k}} = \frac{(u^*)^2}{\sqrt{f k}}
\end{align*}
\]

Note: \( \tilde{z} \) negative downward

Solution for \( \{ \tau_x (\tilde{z}) \}_{\tilde{z}=0} = \tau_s \) \( \{ \tau_y (\tilde{z}) \}_{\tilde{z}=0} = 0 \)

\[
\begin{align*}
u &= U \exp[b \tilde{z}] \sin(b \tilde{z} + \frac{\pi}{4}) \\
v &= -U \exp[b \tilde{z}] \cos(b \tilde{z} + \frac{\pi}{4}) \\
U &= \frac{\tau_x}{\rho_0 \sqrt{f k}} = \frac{(u^*)^2}{\sqrt{f k}}
\end{align*}
\]

Note: \( \tilde{z} \) negative downward
Example: At a latitude of \(45^\circ N\) wind, \(U_{10}\), blows to the north at 10 m/sec. Observations show that the eddy viscosity is \(k = 10^2 \frac{m^2}{\text{sec}}\).

(a) What is the effective Ekman depth?  
(b) What is the magnitude and direction of the surface stress and surface current at \(\ddot{z} = 0\)?  
(c) What is the magnitude and direction of the current and stress at \(\ddot{z} = -5m,-20m\).  
(d) Estimate the turbulent velocity at \(\ddot{z} = 0m, -5m, -20m\).  
(e) How deep would you have to go to be at a depth where the turbulence vanished?

Note: \(f = 2\Omega \sin(\theta) = 2 \frac{2\pi}{24*3600} \text{ rad sec} \sin(45^\circ) = 10^{-4} \text{ rad sec} \)

\[(a) \quad b^{-1} = \sqrt{\frac{2k}{f}} = \sqrt{\frac{2 \cdot 10^{-2}}{10^{-4}}} = 14m \]

\[(b) \quad \tau_s = \rho_0 C_D (U_{10})^2 = 1 \frac{kg}{m^3} 2.5 \cdot 10^{-3} (10 \frac{m}{\text{sec}})^2 = .25 \frac{N}{m^2} \]

\[U = \frac{\tau_y}{\rho_0 \sqrt{fk}} = \frac{.25 \frac{N}{m}}{10^3 \frac{kg}{m^3} \sqrt{10^{-4} \cdot 10^{-2}}} = .25 \frac{m}{\text{sec}}, 45^\circ \text{ to Northeast} \]
(c) \( \tilde{z} = -5m \) \quad u = U \exp[b\tilde{z}] \cos(b\tilde{z} + \frac{\pi}{4}) = .25 \exp[-\frac{5}{14}] \cos(-\frac{5}{14} + \frac{\pi}{4})

= .25 \times .70 \times .91 = .16 \frac{m}{\text{sec}}

v = U \exp[b\tilde{z}] \sin(b\tilde{z} + \frac{\pi}{4}) = .25 \exp[-\frac{5}{14}] \sin(-\frac{5}{14} + \frac{\pi}{4})

= .25 \times .70 \times .41 = .16 \frac{m}{\text{sec}} = .07

\tan(\phi) = \frac{v}{u} \implies \phi = -\frac{5}{14} + \frac{\pi}{4} = -20 + 45 = 25^\circ

\tau_x = \rho_0 k \frac{\partial u}{\partial \tilde{z}} = \rho_0 k Ub \exp(b\tilde{z})[\cos(b\tilde{z} + \frac{\pi}{4}) - \sin(b\tilde{z} + \frac{\pi}{4})]

= \frac{\tau_s}{\sqrt{2}} \exp(b\tilde{z})[\cos(b\tilde{z} + \frac{\pi}{4}) - \sin(b\tilde{z} + \frac{\pi}{4})]

= .25 \frac{\exp(-\frac{5}{14})}{\sqrt{2}}[\cos(-\frac{5}{14} + \frac{\pi}{4}) - \sin(-\frac{5}{14} + \frac{\pi}{4})]

= .061 \frac{N}{m^2}
\[
\tau_y = \rho_0 k \frac{\partial v}{\partial z} = \rho_0 k U b \exp(b\tilde{z}) [\cos(b\tilde{z} + \frac{\pi}{4}) + \sin(b\tilde{z} + \frac{\pi}{4})]
\]

\[
= \frac{\tau_s}{\sqrt{2}} \exp(b\tilde{z}) [\cos(b\tilde{z} + \frac{\pi}{4}) + \sin(b\tilde{z} + \frac{\pi}{4})]
\]

\[
= \frac{.25}{\sqrt{2}} \exp(-\frac{5}{14}) [\cos(-\frac{5}{14} + \frac{\pi}{4}) + \sin(-\frac{5}{14} + \frac{\pi}{4})]
\]

\[
= .163 \frac{N}{m^2}
\]
(c) at $\tilde{z} = 20m$

\[ u = U \exp[b\tilde{z}] \cos(b\tilde{z} + \frac{\pi}{4}) = 0.25 \exp[-\frac{20}{14}] \cos(-\frac{20}{14} + \frac{\pi}{4}) \]

\[ = 0.25 \times 0.23 \times 0.8 = 0.05 \text{ m/sec} \]

\[ v = U \exp[b\tilde{z}] \sin(b\tilde{z} + \frac{\pi}{4}) = 0.25 \exp[-\frac{20}{14}] \sin(-\frac{20}{14} + \frac{\pi}{4}) \]

\[ = 0.25 \times 0.23 \times (-0.6) = -0.03 \text{ m/sec} \]

\[ \tan(\phi) = \frac{v}{u} \Rightarrow \phi = -\frac{20}{14} + \frac{\pi}{4} = -81 + 45 = -36^\circ \]

\[ \tau_x = \rho_0 k \frac{\partial u}{\partial \tilde{z}} = \rho_0 k Ub \exp(b\tilde{z})[\cos(b\tilde{z} + \frac{\pi}{4}) - \sin(b\tilde{z} + \frac{\pi}{4})] \]

\[ = \frac{\tau_s}{\sqrt{2}} \exp(b\tilde{z})[\cos(b\tilde{z} + \frac{\pi}{4}) - \sin(b\tilde{z} + \frac{\pi}{4})] \]

\[ = \frac{0.25}{\sqrt{2}} \exp(-\frac{20}{14})[\cos(-\frac{20}{14} + \frac{\pi}{4}) - \sin(-\frac{20}{14} + \frac{\pi}{4})] \]

\[ = 0.059 \frac{\text{N}}{\text{m}^2} \]
\[ \tau_y = \rho_0 k \frac{\partial v}{\partial z} = \rho_0 k U b \exp(b\tilde{z}) \left[ \cos(b\tilde{z} + \frac{\pi}{4}) + \sin(b\tilde{z} + \frac{\pi}{4}) \right] \]

\[ = \frac{\tau_s}{\sqrt{2}} \exp(b\tilde{z}) \left[ \cos(b\tilde{z} + \frac{\pi}{4}) + \sin(b\tilde{z} + \frac{\pi}{4}) \right] \]

\[ = \frac{.25}{\sqrt{2}} \exp\left(-\frac{20}{14}\right) \left[ \cos\left(-\frac{20}{14} + \frac{\pi}{4}\right) + \sin\left(-\frac{20}{14} + \frac{\pi}{4}\right) \right] \]

\[ = .008 \frac{N}{m^2} \]

(d) \[ u^* = \sqrt{\frac{\tau}{\rho_0}} \]

at \( z=0, \tau = .25 \) \( \Rightarrow u^* = \sqrt{\frac{.25}{1000}} = .016 \frac{m}{sec} \]
at $z=5$, $\tau = \sqrt{\tau_x^2 + \tau_y^2} = \sqrt{.061^2 + .168^2} = .18 \frac{N}{m^2}$

$\Rightarrow u^* = \sqrt{\frac{.18}{1000}} = .013 \frac{m}{\text{sec}}$

at $z=20$, $\tau = \sqrt{\tau_x^2 + \tau_y^2} = \sqrt{.059^2 + .008^2} = .06 \frac{N}{m^2}$

$\Rightarrow u^* = \sqrt{\frac{.06}{1000}} = .0024 \frac{m}{\text{sec}}$

(e) $|\ddot{z}| >> \frac{1}{b}$
\[ \tau_y = \rho_0 k \frac{\partial v}{\partial z} \]

\[ f_u = \frac{1}{\rho} \frac{\partial \tau_y}{\partial z} \]

\[ \tau_x = \rho_0 k \frac{\partial u}{\partial z} \]

\[ -f_v = \frac{1}{\rho} \frac{\partial \tau_x}{\partial z} \]
Approach Using Complex Variables

\[- f v = \frac{1}{\rho} \frac{\partial \tau_x}{\partial z}; \quad \tau_x = \rho_0 k \frac{\partial u}{\partial z} \quad f u = \frac{1}{\rho} \frac{\partial \tau_y}{\partial z}; \quad \tau_y = \rho_0 k \frac{\partial v}{\partial z} \]

\[\Rightarrow - f v = k \frac{\partial^2 u}{\partial z^2} \quad \& \quad f u = k \frac{\partial^2 v}{\partial z^2} \]

Let \( \tilde{u} = u + iv \quad i = \sqrt{-1} = \exp(\frac{\pi}{2} i) \Rightarrow \]

\[i f \tilde{u} = k \frac{\partial^2 \tilde{u}}{\partial z^2} \quad \text{Solution} \quad \tilde{u} = A \exp(-sz) \]

\[s^2 = \frac{if}{k} \quad \Rightarrow \quad s = \sqrt{\frac{if}{k}} = \sqrt{\frac{\exp(\frac{\pi}{2} i) f}{k}} = \exp(\frac{\pi}{4} i) \sqrt{\frac{f}{k}} = (1 + i) b \]

\[b = \sqrt{\frac{f}{2k}} \quad \& \quad |s| = \sqrt{\frac{f}{k}} = \sqrt{2b} \]
Case 1  \( \tau_x(z = 0) = 0; \tau_y(z = 0) \neq 0 \) (Wind to North)

\[
\tilde{\tau} = k \frac{\partial \tilde{u}}{\partial z} = A \exp(\pm sz)
\]

\[
\text{Re}(\frac{\tilde{\tau}}{\rho_0})_{z=0} = 0 \Rightarrow \text{Re}(A) = \text{Re}\{A\exp(\frac{\pi}{4} i) \sqrt{\frac{f}{k}}\} = 0
\]

\[
A = U\exp(\frac{\pi}{4} i)
\]

\[
\tilde{u} = U\exp(\frac{\pi}{4} i) \exp(-sz) \quad u = \text{Re}(\tilde{u}) \quad v = \text{Imag}(\tilde{u})
\]

\[
|\frac{\tilde{\tau}}{\rho_0}| = (u^*)^2 = k |\frac{\partial \tilde{u}}{\partial z}| = U |s| \exp(-bz) = (u^*)^2_{z=0} |\exp(-bz)
\]

\[
U = \frac{(u^*)^2_{z=0}}{\sqrt{fk}}
\]
Case 2 \( \tau_x(z = 0) \neq 0; \tau_y(z = 0) = 0 \) (Wind to East)

\[
\frac{\tilde{\tau}}{\rho_0} = k \frac{\partial \tilde{u}}{\partial z} = A \exp(-sz)
\]

\[
\text{Imag}\left( \frac{\tilde{\tau}}{\rho_0} \right)_{z=0} = 0 \Rightarrow \text{Imag}(A) = \text{Imag}\{A \exp\left( \frac{\pi i}{4} \sqrt{\frac{f}{k}} \right) \} = 0
\]

\[
A = U \exp\left( -\frac{\pi i}{4} \right)
\]

\[
\tilde{u} = U \exp\left( -\frac{\pi i}{4} \right) \exp(-sz) \quad u = \text{Re}(\tilde{u}) \quad v = \text{Imag}(\tilde{u})
\]

\[
\left| \frac{\tilde{\tau}}{\rho_0} \right| = (u^*)^2 = k \left| \frac{\partial \tilde{u}}{\partial z} \right| = U \left| s \right| \exp(-b z) = (u^*)^2_{z=0} \left| \exp(-b z) \right|
\]

\[
U = \frac{(u^*)^2_{z=0}}{\sqrt{fk}}
\]
Ekman Dynamics: Bottom Layer

North

Sea surface slopes
Upward to south

\[ \frac{\partial p}{\partial y} < 0 \]

Deep Ocean

\[ u_g = -\frac{1}{f \rho_0} \frac{\partial p}{\partial y} \]

Eddy Viscosity

\[ \tau_x = k \frac{\partial u}{\partial z} \]
\[ \tau_y = k \frac{\partial v}{\partial z} \]

Assumptions

Steady State
Upper Ocean Geostrophic
Deep Ocean

East
Case 1: Pressure Gradient to North

Equations

\[-fv = k \frac{\partial^2 u}{\partial z^2}\]

\[fu = fu_g + k \frac{\partial^2 v}{\partial z^2}\]

Solutions

\[u = u_g \left[1 - \exp(-bz) \cos(bz)\right]\]

\[v = u_g \exp(-bz) \sin(bz)\]

Note: \(z\) positive upward

\[b = \sqrt{\frac{f}{2k}} \quad bH >> 1\]

View Looking Down

\[\tau_b = \rho_0 u_g \sqrt{fk}\]
Case 2: Pressure Gradient to West

Equations

\[-fv = -fv_g + k \frac{\partial^2 u}{\partial z^2}\]
\[fu = k \frac{\partial^2 v}{\partial z^2}\]

Solutions

\[u = -v_g \exp(-bz) \sin(bz)\]
\[v = v_g [1 - \exp(-bz) \cos(bz)]\]
\[b = \sqrt{\frac{f}{2k}} \quad bH \gg 1\]

View Looking Down

\[
\begin{align*}
\vec{u} & \quad \text{West} \\
\vec{v}_g & \quad \text{South} \\
\vec{\tau}_b & \quad \text{East} \\
\end{align*}
\]

\[
\tau_b = \rho_0 v_g \sqrt{fk}
\]
Example: At a latitude of $45^\circ N$ under no wind conditions the geostrophic current is observed to be .5 m/sec to the east. Observations show that the eddy viscosity is $k = 2 \cdot 10^{-2} \frac{m^2}{sec}$

(a) What is the magnitude and direction of the sea surface slope? (b) What is the effective Ekman depth? (c) What is the magnitude and direction of the stress and current on the bottom, at $z = 0$, at $z = 20$m, at $z = 40$m? (d) Estimate the turbulent velocity at $z = 0$, 5m, 20 m. (e) How shallow would you have to go to be at a depth where the turbulence vanished?

\[ p = \rho_0 g (z + \zeta) \]

\[ \frac{\partial p}{\partial y} = \rho_0 g \alpha \]

\[ fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \Rightarrow \alpha = -\frac{fu}{g} = \frac{10^{-4} \text{ sec}^{-1} \cdot 0.5 \frac{m}{\text{sec}}}{10 \frac{m}{\text{sec}^2}} = -5 \cdot 10^{-6} \]

\[ b^{-1} = \sqrt{\frac{2k}{f}} = \sqrt{\frac{2 \cdot 2 \cdot 10^{-2}}{10^{-4}}} = 20 \text{m} \]
(c) \[ \tau_b = \rho_0 u_g \sqrt{fk} = 10^3 \cdot 5 \cdot \sqrt{10^{-4} \cdot 2 \cdot 10^{-2}} = 0.7 \frac{N}{m^2} \]

to the northeast

\[ u = v = 0 \quad \text{at } z = 0 \]

\[ u = u_g [1 - \exp(-bz) \cos(bz)] \]

\[ v = u_g \exp(-bz) \sin(bz) \]

\[ \tau_x = k \frac{\partial u}{\partial z} = ku_g b \exp(-bz)[\cos(bz) + \sin(bz)] \]

\[ = \frac{\tau_b}{\sqrt{2}} \exp(-bz)[\cos(bz) + \sin(bz)] \]

\[ \tau_y = k \frac{\partial v}{\partial z} = ku_g b \exp(-bz)[\cos(bz) - \sin(bz)] \]

\[ = \frac{\tau_b}{\sqrt{2}} \exp(-bz)[\cos(bz) - \sin(bz)] \]
At \( z = 20 \text{ m} \)

\[
\begin{align*}
\mathbf{u} &= u_g [1 - \exp(-bz) \cos(bz)] = .5(1 - \exp(-\frac{20}{20}) \cos(1)) = 0.4 \frac{m}{\text{sec}} \\
\mathbf{v} &= u_g \exp(-bz) \sin(bz) = .5(\exp(-\frac{20}{20}) \sin(1)) = 0.15 \frac{m}{\text{sec}} \quad \text{(where } b = 0.7) \\
\tau_x &= \frac{\tau_b}{\sqrt{2}} \exp(-bz)[\cos(bz) + \sin(bz)] = \frac{0.7}{\sqrt{2}} \exp(-1)[\cos(1) + \sin(1)] \\
&= 0.25 \frac{N}{m^2} \\
\tau_y &= \frac{\tau_b}{\sqrt{2}} \exp(-bz)[\cos(bz) - \sin(bz)] = \frac{0.7}{\sqrt{2}} \exp(-1)[\cos(1) - \sin(1)] \\
&= -0.05 \frac{N}{m^2}
\end{align*}
\]
At \(z = 40\) m

\[
u = u_g \left[1 - \exp(-b z) \cos(b z)\right] = 0.5 \left[1 - \exp(-2) \cos(2)\right] = 0.52 \text{ m/sec}
\]

\[
v = u_g \exp(-b z) \sin(b z) = 0.5 \exp(-2) \sin(2) = 0.056 \text{ m/sec}
\]

\[
\tau_x = \frac{\tau_b}{\sqrt{2}} \exp(-b z) \left[\cos(b z) + \sin(b z)\right]= \frac{0.7}{\sqrt{2}} \exp(-2) \left[\cos(2) + \sin(2)\right] = 0.033 \text{ N/m}^2
\]

\[
\tau_y = \frac{\tau_b}{\sqrt{2}} \exp(-b z) \left[\cos(b z) - \sin(b z)\right]= \frac{0.7}{\sqrt{2}} \exp(-2) \left[\cos(2) - \sin(2)\right] = -0.09 \text{ N/m}^2
\]

At \(z = 0, 20, 40\) m

\[
\tau = 0.5, 0.25, 0.09 \text{ N/m}^2
\]

\[
u^* = \sqrt{\frac{\tau}{\rho_0}} = 0.022, 0.0158, 0.0095 \text{ m/sec}
\]
Force Balance

Geostrophy
\[ f u_g = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \]

Pressure Gradient Force
\[ F_p \]

Pressure Gradient Force
\[ \nabla p \]

Coriolis Force
\[ F_{Co} \]

Bottom Stress \( \tau_B \)
\[ |\tau_B| = \rho_0 u_g \sqrt{kf} \]

Ekman Dynamics
\[ -fv = \frac{1}{\rho} \frac{\partial \tau_x}{\partial z} \]
\[ fu = -\frac{1}{\rho} (\frac{\partial p}{\partial y}) + \frac{1}{\rho} \frac{\partial \tau_y}{\partial z} \]

Note: Forces are obtained by a vertical integral of above equations.
Let $v = (\mathbf{u} - \mathbf{u}_g)$.

$$f \times (\mathbf{u} - \mathbf{u}_g) = \frac{1}{\rho_0} \frac{\partial \tau}{\partial z} \Rightarrow f\mathbf{u}_g = -\frac{1}{\rho_0} \nabla p$$

Let $\mathbf{v} = (\mathbf{u} - \mathbf{u}_g)$.

$$\frac{1}{\rho} \int_0^\infty dz \frac{\partial \tau}{\partial z} = -\frac{1}{\rho} \tau_{z=0} = \int_0^\infty dz \mathbf{f} \times (\mathbf{u} - \mathbf{u}_g)$$

$$\frac{\tau_{z=0}}{\rho} = fV \Rightarrow V = u_g \sqrt{\frac{k}{f}} = \frac{u_g b^{-1}}{\sqrt{2}}$$