Momentum Equations in a Fluid

Total Force acting on a body = mass times its acceleration  \[ F = ma \]

The forces on a cube of water

- Pressure difference (PD)
- Coriolis Force (Co)
- Friction (Fr)
- Weight (W)

\[
a_i = \frac{F_i}{m} \Rightarrow \frac{du_i}{dt} = \frac{1}{\rho} (PD_i + Co_i + Fr_i + W_i)
\]

Note the subscript "i".

\[
\rho = \frac{M}{V}
\]

density of water
Navier Stokes Equation with Gravity and Coriolis

\[ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = - \frac{1}{\rho_0} \nabla p - \vec{b} + \nu \nabla^2 \vec{u} \]

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \varepsilon_{ijk} f_j u_k = - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - b_i + \nu \frac{\partial^2 u_i}{\partial x_i^2} \]

\[ f_i = 2\Omega \sin(\theta) \delta_{i3} \quad 2\Omega = 1.5 \times 10^{-4} \text{ sec}^{-1}; \theta \text{ Latitude} \]

\[ b_i = \frac{\Delta \rho}{\rho_0} g \delta_{i3} \]

\[ \nabla = \frac{\partial}{\partial x_i} \quad \nabla^2 = \frac{\partial^2}{\partial x_i^2} \]
Navier Stokes Equation with Gravity and Coriolis Component Form

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0} \left( \frac{\partial p}{\partial x} \right) + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho} \left( \frac{\partial p}{\partial y} \right) + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \left( \frac{\partial p}{\partial z} \right) - b + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align*}
\]
Generalizing Conservation of Mass flow: 

Continuity Equation

\[ \sum u_i A_i = 0 \]

sum of velocity times area over all 6 faces = 0

Continuity (Incompressibility) Equation

\[ \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]
\[ \frac{\text{Vorticity}}{2} = \text{angular velocity} = \frac{\text{velocity}}{\text{radius}} \]

\[ \frac{\omega}{2} = \alpha = \frac{v}{r} \]

But vorticity is a vector! (why?)

\[ \vec{\omega} = \nabla \times \vec{u} \]

Component form

\[ \omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \]

\[ \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \]

\[ \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

Most important for Large scale flow \( \sim 20 \text{km} \)
Total (absolute) Vorticity
\[ \vec{\Omega} = \vec{f} + \vec{\omega} \]

Vorticity Equation

\[ \frac{d\vec{\Omega}}{dt} = \vec{\Omega} \cdot \nabla \hat{u} + v \nabla^2 \vec{\Omega} \]

- Stretching Term
- Friction Term
Potential Vorticity (Ertel)
(Angular momentum per unit volume)

\[ q = \tilde{\Omega} \cdot \nabla \rho \]

Using \( \frac{d\rho}{dt} = 0 \) Assuming Frictionless flow
& Assuming Frictionless flow \( \frac{d\tilde{\Omega}}{dt} = \tilde{\Omega} \cdot \nabla \vec{u} \)

Conservation of Potential Vorticity
\[ \frac{dq}{dt} = 0 \]
Conservation of Potential Vorticity = Conservation of Angular Momentum + Conservation of Mass Following a Fluid Parcel (Case of \( f = 0 \))

\[ \omega_1 I_1 = \omega_2 I_2 \]

\( \omega \) vorticity

\( I = Amr^2 \) moment of inertia

\[ I\omega = (Amr^2)\omega \]

Using \( I_1 \omega_1 = I_1 \omega_1 \) & \( V_1 = V_2 \)

\[ \Rightarrow \frac{\omega_1}{H_1} = \frac{\omega_2}{H_2} \]
Vorticity

Concept of Conservation of angular momentum

Definition of Vorticity
\[ \Omega = 2 \times \text{angular velocity} \]
\[ \Omega \text{ units of } 1/\text{sec} \]
Conservation of Potential Vorticity $f \neq 0$

\[
\frac{\omega_1 + f_1}{H_1} = \frac{\omega_2 + f_2}{H_2}
\]

\[
\frac{d}{dt} \left\{ \frac{\omega + f}{H} \right\} = 0
\]

\[
\frac{d}{dt} \{(\omega + f)N^2\} = 0
\]

**Terminology**

- $\omega$ = relative vorticity
- $f = 2\Omega \sin(\theta)$ = planetary vorticity
- $\omega + f$ = total vorticity
- $\Pi = \frac{\omega + f}{H}$ = Potential vorticity (Common Definition)
Homework: von Arx (1962) has suggested that to understand conservation of potential vorticity we consider what happens to a barrel of water as it moves around the earth. Take the shaded circles as the initial rest position (no relative vorticity) of a barrel of water. Assume that the shape and volume of the barrel is constant, i.e. $H$, $r$ constant. Using the concept of conservation of potential vorticity explain what happens to the relative vorticity and its direction of rotation (clockwise, counterclockwise) as it moves along paths #1, 2, 3, 4, 5, 6, 7.
Potential Flow

\[ \vec{\omega} = 0 \Rightarrow \vec{u} = \nabla \phi \]

Incompressible Fluid

\[ \nabla \cdot \vec{u} = 0 \Rightarrow \nabla^2 \phi = 0 \]

\[ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial w} \]

Note: Turbulence always has vorticity
2D Incompressible Flow
Concept of a Stream Function $\psi$

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies
\]

\[
u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}
\]
Fundamental Equations

\[
\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} + \tilde{f} \times \tilde{u} = -\frac{1}{\rho} \nabla p - \tilde{b} + \nu \nabla^2 \tilde{u}
\]

\[
\nabla \cdot \tilde{u} = 0
\]

\[
\frac{d \tilde{\Omega}}{dt} = \tilde{\Omega} \cdot \nabla \tilde{u} + \nu \nabla^2 \tilde{\Omega}
\]

\[
\frac{d \rho}{dt} = 0
\]