Chapter 3

Kinematics
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1. Introduction

Kinematics is the branch of mechanics that deals with quantities involving space and time only. It treats variables such as displacement, velocity, acceleration, deformation, and rotation of fluid elements without referring to the forces responsible for such a motion. Kinematics therefore essentially describes the "appearance" of a motion. Some important kinematical concepts are described in this chapter. The forces are considered when one deals with the dynamics of the motion, which will be discussed in later chapters.

A few remarks should be made about the notation used in this chapter and throughout the rest of the book. The convention followed in Chapter 2, namely, that vectors are denoted by lowercase letters and higher-order tensors are denoted by uppercase letters, is no longer followed. Henceforth, the number of subscripts will specify the order of a tensor. The Cartesian coordinate directions are denoted by \((x, y, z)\), and the corresponding velocity components are denoted by \((u, v, w)\). When using tensor expressions, the Cartesian directions are denoted alternatively by \((x_1, x_2, x_3)\), with the corresponding velocity components \((u_1, u_2, u_3)\). Plane polar coordinates are denoted by \((r, \theta)\), with \(u_r\) and \(u_\theta\) the corresponding velocity components (Figure 3.1a). Cylindrical polar coordinates are denoted by \((R, \varphi, z)\), with \((u_R, u_\varphi, u_z)\) the corresponding velocity components (Figure 3.1b). Spherical polar coordinates are denoted by \((r, \theta, \varphi)\), with \((u_r, u_\theta, u_\varphi)\) the corresponding velocity components (Figure 3.1c). The method of conversion from Cartesian to plane polar coordinates is illustrated in Section 14 of this chapter.

2. Lagrangian and Eulerian Specifications

There are two ways of describing a fluid motion. In the Lagrangian description, one essentially follows the history of individual fluid particles (Figure 3.2). Consequently, the two independent variables are taken as time and a label for fluid particles. The label can be conveniently taken as the position vector \(\mathbf{x}_0\) of the particle at some reference time \(t = 0\). In this description, then, any flow variable \(F\) is expressed as \(F(\mathbf{x}_0, t)\). In particular, the particle position is written as \(\mathbf{x}(\mathbf{x}_0, t)\), which represents the location at \(t\) of a particle whose position was \(\mathbf{x}_0\) at \(t = 0\).

In the Eulerian description, one concentrates on what happens at a spatial point \(\mathbf{x}\), so that the independent variables are taken as \(\mathbf{x}\) and \(t\). That is, a flow variable is written as \(F(\mathbf{x}, t)\)
The velocity and acceleration of a fluid particle in the Lagrangian description are simply the partial time derivatives

\[ u_t = \frac{\partial x_j}{\partial t}, \quad a_t = \frac{\partial u_j}{\partial t} = \frac{\partial^2 x_j}{\partial t^2}. \]  

(3.1)

as the particle identity is kept constant during the differentiation. In the Eulerian description, however, the partial derivative \( \partial / \partial t \) gives only the local rate of change at a point \( x \), and is not the total rate of change seen by a fluid particle. Additional terms are needed to form derivatives following a particle in the Eulerian description, as explained in the next section.

The Eulerian specification is used in most problems of fluid flows. The Lagrangian description is used occasionally when we are interested in finding particle paths of fixed identity; examples can be found in Chapters 7 and 13.

3. Material Derivative

Let \( F \) be any field variable, such as temperature, velocity, or stress. Employing Eulerian coordinates \((x, y, z, t)\), we seek to calculate the rate of change of \( F \) at each point following a particle of fixed identity. The task is therefore to represent a concept essentially Lagrangian in nature in Eulerian language.

For arbitrary and independent increments \( dx \) and \( dt \), the increment in \( F(x, t) \) is

\[ dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x_i} dx_i, \]  

(3.2)

where a summation over the repeated index \( i \) is implied. Now assume that the increments are not arbitrary, but those associated with following a particle of fixed identity.

The increments \( dx \) and \( dt \) are then no longer independent, but are related to the velocity components by the three relations represented by

\[ dx_i = u_i dt. \]

Substitution into Eq. (3.2) gives

\[ \frac{dF}{dt} = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}. \]  

(3.3)

The notation \( dF / dt \), however, is too general. In order to emphasize the fact that the time derivative is carried out as one follows a particle, a special notation \( D / Dt \) is frequently used in place of \( d / dt \) in fluid mechanics. Accordingly, Eq. (3.3) is written as

\[ \frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}. \]  

(3.4)

The total rate of change \( D / Dt \) is generally called the material derivative (also called the substantial derivative, or particle derivative) to emphasize the fact that the derivative is taken following a fluid element. It is made of two parts: \( \partial F / \partial t \) is the local rate of change of \( F \) at a given point, and is zero for steady flows. The second part \( u_i \partial F / \partial x_i \) is called the advective derivative, because it is the change in \( F \) as a result of advection of the particle from one location to another where the value of \( F \) is different. (In this book, the movement of fluid from one place to place is called “advection.” Engineering texts generally call it “convection.” However, we shall reserve the term convection to describe heat transport by fluid movements.)

In vector notation, Eq. (3.4) is written as

\[ \frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F. \]  

(3.5)

The scalar product \( \mathbf{u} \cdot \nabla F \) is the magnitude of \( \mathbf{u} \) times the component of \( \nabla F \) in the direction of \( \mathbf{u} \). It is customary to denote the magnitude of the velocity vector \( \mathbf{u} \) by \( q \). Equation (3.5) can then be written in scalar notation as

\[ \frac{DF}{Dt} = \frac{\partial F}{\partial t} + q \frac{\partial F}{\partial s}, \]  

(3.6)

where the “streamline coordinate” \( s \) points along the local direction of \( \mathbf{u} \) (Figure 3.3).

4. Streamline, Path Line, and Streak Line

At an instant of time, there is at every point a velocity vector with a definite direction. The instantaneous curves that are everywhere tangent to the direction field are called the streamlines of flow. For unsteady flows the streamline pattern changes with time. Let \( ds = (dx, dy, dz) \) be an element of arc length along a streamline (Figure 3.4),
and let \( \mathbf{u} = (u, v, w) \) be the local velocity vector. Then by definition

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w},
\]

along a streamline. If the velocity components are known as a function of time, then Eq. (3.7) can be integrated to find the equation of the streamline. It is easy to show that Eq. (3.7) corresponds to \( \mathbf{u} \times ds = 0 \). All streamlines passing through any closed curve \( C \) at some time form a tube, which is called a streamtube (Figure 3.5). No fluid can cross the streamtube because the velocity vector is tangent to this surface.

In experimental fluid mechanics, the concept of path line is important. The path line is the trajectory of a fluid particle of fixed identity over a period of time. Path lines and streamlines are identical in a steady flow, but not in an unsteady flow. Consider the flow around a body moving from right to left in a fluid that is stationary at an infinite distance from the body (Figure 3.6). The flow pattern observed by a stationary observer (that is, an observer stationary with respect to the undisturbed fluid) changes with time, so that to the observer this is an unsteady flow. The streamlines in front of and behind the body are essentially directed forward as the body pushes forward, and those on the two sides are directed laterally. The path line (shown dashed in

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**Figure 3.3** Streamline coordinates \((x, n)\).

**Figure 3.4** Streamline.

**Figure 3.5** Streamtube.

**Figure 3.6** Several streamlines and a path line due to a moving body.
Figure 3.6) of the particle that is now at point P therefore loops outward and forward again as the body passes by.

The streamlines and path lines of Figure 3.6 can be visualized in an experiment by suspending aluminum or other reflecting materials on the fluid surface, illuminated by a source of light. Suppose that the entire fluid is covered with such particles, and a brief time exposure is made. The photograph then shows short dashes, which indicate the instantaneous directions of particle movement. Smooth curves drawn through these dashes constitute the instantaneous streamlines. Now suppose that only a few particles are introduced, and that they are photographed with the shutter open for a long time. Then the photograph shows the paths of a few individual particles, that is, their path lines.

A streak line is another concept in flow visualization experiments. It is defined as the current location of all fluid particles that have passed through a fixed spatial point at a succession of previous times. It is determined by injecting dye or smoke at a fixed point for an interval of time. In steady flow the streamlines, path lines, and streak lines all coincide.

5. Reference Frame and Streamline Pattern

A flow that is steady in one reference frame is not necessarily so in another. Consider the flow past a ship moving at a steady velocity \( U \), with the frame of reference (that is, the observer) attached to the river bank (Figure 3.7a). To this observer the local flow characteristics appear to change with time, and thus appear to be unsteady. If, on the other hand, the observer is standing on the ship, the flow pattern is steady (Figure 3.7b). The steady flow pattern can be obtained from the unsteady pattern of Figure 3.7a by superposing on the latter a velocity \( U \) to the right. This causes the ship to come to a halt and the river to move with velocity \( U \) at infinity. It follows that any velocity vector \( u \) in Figure 3.7b is obtained by adding the corresponding velocity vector \( u' \) of Figure 3.7a and the free stream velocity vector \( U \).

6. Linear Strain Rate

A study of the dynamics of fluid flows involves determination of the forces on an element, which depend on the amount and nature of its deformation, or strain. The deformation of a fluid is similar to that of a solid, where one defines normal strain as

![Figure 3.7](image)

Flow past a ship with respect to two observers: (a) observer on river bank; (b) observer on ship.

![Figure 3.8](image)

Figure 3.8  Linear strain rate. Here, \( AB' = AB + BB' - AA' \).

the change in length per unit length of a linear element, and shear strain as change of a 90° angle. Analogous quantities are defined in a fluid flow, the basic difference being that one defines strain rates in a fluid because it continues to deform.

Consider first the linear or normal strain rate of a fluid element in the \( x_1 \) direction (Figure 3.8). The rate of change of length per unit length is

\[
\frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1) = \frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1) = \frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1) = \frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1) = \frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1)
\]

The material derivative symbol \( D/\Delta t \) has been used because we have implicitly followed a fluid particle. In general, the linear strain rate in the \( \alpha \) direction is

\[
\frac{\partial u_\alpha}{\partial x_\beta}
\]

where no summation over the repeated index \( \beta \) is implied. Greek symbols such as \( \alpha \) and \( \beta \) are commonly used when the summation convention is violated.

The sum of the linear strain rates in the three mutually orthogonal directions gives the rate of change of volume per unit volume, called the volumetric strain rate (also called the bulk strain rate). To see this, consider a fluid element of sides \( \delta x_1 \), \( \delta x_2 \), and \( \delta x_3 \). Defining \( \delta V = \delta x_1 \delta x_2 \delta x_3 \), the volumetric strain rate is

\[
\frac{1}{\delta V} \frac{D}{Dt} (\delta V) = \frac{1}{\delta x_1} \frac{D}{Dt} (\delta x_1) + \frac{1}{\delta x_2} \frac{D}{Dt} (\delta x_2) + \frac{1}{\delta x_3} \frac{D}{Dt} (\delta x_3),
\]

that is,

\[
\frac{1}{\delta V} \frac{D}{Dt} (\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.
\]

The quantity \( \partial u_\alpha/\partial x_\beta \) is the sum of the diagonal terms of the velocity gradient tensor \( \partial u_\alpha/\partial x_\beta \). As a scalar, it is invariant with respect to rotation of coordinates. Equation (3.9) will be used later in deriving the law of conservation of mass.
average rotation rate of the two lines is calculated; it is easy to show that this average is independent of the orientation of the line pair. To avoid the appearance of certain factors of \(2\) in the final expressions, it is generally customary to deal with twice the angular velocity, which is called the vorticity of the element.

Consider the two perpendicular line elements of Figure 3.9. The angular velocities of line elements about the \(x_3\) axis are \(d\beta/dt\) and \(-d\alpha/dt\), so that the average is \(\frac{1}{2}(-d\alpha/dt + d\beta/dt)\). The vorticity of the element about the \(x_3\) axis is therefore twice this average, as given by

\[
\omega_3 = \frac{1}{2} \left( -\frac{\partial u_1}{\partial x_2} \delta x_2 dt + \frac{\partial u_2}{\partial x_1} \delta x_1 dt \right)
\]

From the definition of curl of a vector (see Eqs. 2.24 and 2.25), it follows that the vorticity vector of a fluid element is related to the velocity vector by

\[
\omega = \nabla \times \mathbf{u} \quad \text{or} \quad \omega_i = e_{ijk} \frac{\partial u_j}{\partial x_k},
\]

whose components are

\[
\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.
\]

A fluid motion is called irrotational if \(\omega = 0\), which would require

\[
\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \quad i \neq j.
\]

In irrotational flows, the velocity vector can be written as the gradient of a scalar function \(\phi(x, t)\). This is because the assumption

\[
u_i = \frac{\partial \phi}{\partial x_i}
\]

satisfies the condition of irrotationality (3.14).

Related to the concept of vorticity is the concept of circulation. The circulation \(\Gamma\) around a closed contour \(C\) (Figure 3.10) is defined as the line integral of the tangential component of velocity and is given by

\[
\Gamma = \oint_C \mathbf{u} \cdot ds,
\]

where \(ds\) is an element of contour, and the loop through the integral sign signifies that the contour is closed. The loop will be omitted frequently because it is understood
that such line integrals are taken along closed contours called circuits. Then Stokes' theorem (Chapter 2, Section 14) states that

\[ \int_C \mathbf{u} \cdot d\mathbf{s} = \int_A (\text{curl } \mathbf{u}) \cdot d\mathbf{A} \]  

(3.17)

which says that the line integral of \( \mathbf{u} \) around a closed curve \( C \) is equal to the "flux" of curl \( \mathbf{u} \) through an arbitrary surface \( A \) bounded by \( C \). (The word "flux" is generally used to mean the integral of a vector field normal to a surface. [See Eq. (2.32), where the integral written is the net outward flux of the vector field \( \mathbf{Q} \).]) Using the definitions of vorticity and circulation, Stokes' theorem, Eq. (3.17), can be written as

\[ \Gamma = \int_A \omega \cdot d\mathbf{A}. \]  

(3.18)

Thus, the circulation around a closed curve is equal to the surface integral of the vorticity, which we can call the flux of vorticity. Equivalently, the vorticity at a point equals the circulation per unit area. That follows directly from the definition of curl as the limit of the circulation integral. (See Eq. (2.55) of Chapter 2.)

9. Relative Motion near a Point: Principal Axes

The preceding two sections have shown that fluid particles deform and rotate. In this section we shall formally show that the relative motion between two neighboring points can be written as the sum of the motion due to local rotation, plus the motion due to local deformation.

Let \( \mathbf{u}(x, \mathbf{r}) \) be the velocity at point \( O \) (position vector \( \mathbf{x} \)), and let \( \mathbf{u} + d\mathbf{u} \) be the velocity at the same time at a neighboring point \( P \) (position vector \( \mathbf{x} + d\mathbf{x} \); see

Figure 3.11 Velocity vectors at two neighboring points \( O \) and \( P \).

Figure 3.11). The relative velocity at time \( t \) is given by

\[ du_i = \frac{\partial u_i}{\partial x_j} \, dx_j. \]  

(3.19)

which stands for three relations such as

\[ du_1 = \frac{\partial u_1}{\partial x_1} \, dx_1 + \frac{\partial u_1}{\partial x_2} \, dx_2 + \frac{\partial u_1}{\partial x_3} \, dx_3. \]

The term \( \partial u_i / \partial x_j \) in Eq. (3.19) is the velocity gradient tensor. It can be decomposed into symmetric and antisymmetric parts as follows:

\[ \frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \]

which can be written as

\[ \frac{\partial u_i}{\partial x_j} = e_{ij} + \frac{1}{2} r_{ij}. \]  

(3.20)

where \( e_{ij} \) is the strain rate tensor defined in Eq. (3.11), and

\[ r_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}, \]

(3.21)

is called the rotation tensor. As \( r_{ij} \) is antisymmetric, its diagonal terms are zero and the off-diagonal terms are equal and opposite. It therefore has three independent
elements, namely, \( r_{13}, r_{23}, \) and \( r_{32} \). Comparing Eqs. (3.13) and (3.21), we can see that \( r_{13} = \omega_3, r_{23} = \omega_1, \) and \( r_{32} = \omega_2 \). Thus the rotation tensor can be written in terms of the components of the vorticity vector as

\[
\mathbf{r} = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}.
\tag{3.22}
\]

Each antisymmetric tensor in fact can be associated with a vector as discussed in Chapter 2, Section 11. In the present case, the rotation tensor can be written in terms of the vorticity vector as

\[
r_{ij} = -\epsilon_{ijk} \omega_k \tag{3.23}
\]

This can be verified by taking various components of Eq. (3.23) and comparing them with Eq. (3.22). For example, Eq. (3.23) gives \( r_{12} = -\epsilon_{123} \omega_3 = -\epsilon_{132} \omega_1 = -\omega_3 \), which agrees with Eq. (3.22). Equation (3.23) also appeared as Eq. (2.27).

Substitution of Eqs. (3.20) and (3.23) into Eq. (3.19) gives

\[
du_i = \epsilon_{ij} dx_j - \frac{1}{2} \epsilon_{ijl} \omega_l dx_j,
\]

which can be written as

\[
du_i = \epsilon_{ij} dx_j + \frac{1}{2} (\omega \times dx),
\tag{3.24}
\]

In the preceding, we have noted that \( \epsilon_{ijl} \omega_l dx_j \) is the \( i \)-component of the cross product \(-\omega \times dx\). (See the definition of cross product in Eq. (2.21).) The meaning of the second term in Eq. (3.24) is evident. We know that the velocity at a distance \( x \) from the axis of rotation of a body rotating rigidly at angular velocity \( \Omega \) is \( \Omega \times x \). The second term in Eq. (3.24) therefore represents the relative velocity at point \( P \) due to rotation of the element at angular velocity \( \omega / 2 \). (Recall that the angular velocity is half the vorticity \( \omega \).)

The first term in Eq. (3.24) is the relative velocity due only to deformation of the element. The deformation becomes particularly simple in a coordinate system coinciding with the principal axes of the strain rate tensor. The components of \( \epsilon \) change as the coordinate system is rotated. For a particular orientation of the coordinate system, a symmetric tensor has only diagonal components; these are called the principal axes of the tensor (see Chapter 2, Section 12 and Example 2.2). Denoting the variables in the principal coordinate system by an overbar (Figure 3.12), the first part of Eq. (3.24) can be written as the matrix product

\[
d\bar{u} = \epsilon \cdot d\bar{x} = \begin{bmatrix}
\bar{e}_{11} & 0 & 0 \\
0 & \bar{e}_{22} & 0 \\
0 & 0 & \bar{e}_{33}
\end{bmatrix}
\begin{bmatrix}
d\bar{x}_1 \\
d\bar{x}_2 \\
d\bar{x}_3
\end{bmatrix}.
\tag{3.25}
\]

Here, \( \bar{e}_{11}, \bar{e}_{22}, \) and \( \bar{e}_{33} \) are the diagonal components of \( \epsilon \) in the principal coordinate system and are called the eigenvalues of \( \epsilon \). The three components of Eq. (3.25) are

\[
d\bar{u}_1 = \bar{e}_{11} d\bar{x}_1, \quad d\bar{u}_2 = \bar{e}_{22} d\bar{x}_2, \quad d\bar{u}_3 = \bar{e}_{33} d\bar{x}_3.
\tag{3.26}
\]

Consider the significance of the first of equations (3.26), namely, \( d\bar{u}_1 = \bar{e}_{11} d\bar{x}_1 \) (Figure 3.12). If \( \bar{e}_{11} \) is positive, then this equation shows that point \( P \) is moving away from \( O \) in the \( \bar{x}_1 \) direction at a rate proportional to the distance \( d\bar{x}_1 \). Considering all points on the surface of a sphere, the motion of \( P \) in the \( \bar{x}_1 \) direction is therefore the maximum when \( P \) coincides with \( M \) (where \( d\bar{x}_1 \) is the maximum) and is zero when \( P \) coincides with \( N \). (In Figure 3.12 we have illustrated a case where \( \bar{e}_{11} > 0 \) and \( \bar{e}_{22} < 0 \); the deformation in the \( \bar{x}_2 \) direction cannot, of course, be shown in this figure.) In a small interval of time, a spherical fluid element around \( O \) therefore becomes an ellipsoid whose axes are the principal axes of the strain tensor \( \epsilon \).

Summary: The relative velocity in the neighborhood of a point can be divided into two parts. One part is due to the angular velocity of the element, and the other part is due to deformation. A spherical element deforms to an ellipsoid whose axes coincide with the principal axes of the local strain rate tensor.

10. Kinematic Considerations of Parallel Shear Flows

In this section we shall consider the rotation and deformation of fluid elements in the parallel shear flow \( \mathbf{u} = [u_1(x_2), 0, 0] \) shown in Figure 3.13. Let us denote the velocity gradient by \( \gamma(x_2) \equiv du_1/dx_2 \). From Eq. (3.13), the only nonzero component of vorticity is \( \omega_3 = -\gamma \). In Figure 3.13, the angular velocity of line element \( AB \) is
11. Kinematic Considerations of Vortex Flows

Flows in circular paths are called vortex flows, some basic forms of which are described in what follows.

Solid-Body Rotation

Consider first the case in which the velocity is proportional to the radius of the streamlines. Such a flow can be generated by steadily rotating a cylindrical tank containing a viscous fluid and waiting until the transients die out. Using polar coordinates $\left( r, \theta \right)$, the velocity in such a flow is

$$u_\theta = \omega_0 r,$$
$$u_r = 0,$$  \hspace{1cm} (3.27)

where $\omega_0$ is a constant equal to the angular velocity of revolution of each particle about the origin (Figure 3.14). We shall see shortly that $\omega_0$ is also equal to the angular speed of rotation of each particle about its own center. The vorticity components of a fluid element in polar coordinates are given in Appendix B. The component about the z-axis is

$$\omega_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r u_\theta \right) - \frac{1}{r} \frac{\partial}{\partial \theta} u_r = 2\omega_0,$$  \hspace{1cm} (3.28)

where we have used the velocity distribution equation (3.27). This shows that the angular velocity of each fluid element about its own center is a constant and equal to $\omega_0$. This is evident in Figure 3.14, which shows the location of element ABCD at two successive times. It is seen that the two mutually perpendicular fluid lines AD and AB both rotate counterclockwise (about the center of the element) with speed $\omega_0$.

Figure 3.14: Solid-body rotation. Fluid elements are spinning about their own centers while they revolve around the origin. There is no deformation of the elements.
The time period for one rotation of the particle about its own center equals the time period for one revolution around the origin. It is also clear that the deformation of the fluid elements in this flow is zero, as each fluid particle retains its location relative to other particles. A flow defined by \( u_\theta = \omega_0 r \) is called a solid-body rotation as the fluid elements behave as in a rigid, rotating solid.

The circulation around a circuit of radius \( r \) in this flow is

\[
\Gamma = \int ds \cdot \mathbf{u} = \int_0^{2\pi} u_\theta r \; d\theta = 2\pi r u_\theta = 2\pi r^2 \omega_0, \tag{3.29}
\]

which shows that circulation equals vorticity \( 2\omega_0 \) times area. It is easy to show (Exercise 12) that this is true of any contour in the fluid, regardless of whether or not it contains the center.

**Irrotational Vortex**

Circular streamlines, however, do not imply that a flow should have vorticity everywhere. Consider the flow around circular paths in which the velocity vector is tangential and is inversely proportional to the radius of the streamline. That is,

\[
u_{\theta} = \frac{C}{r}, \quad u_r = 0. \tag{3.30}
\]

Using Eq. (3.28), the vorticity at any point in the flow is

\[
\omega_\theta = \frac{0}{r}. \tag{3.31}
\]

This shows that the vorticity is zero everywhere except at the origin, where it cannot be determined from this expression. However, the vorticity at the origin can be determined by considering the circulation around a circuit enclosing the origin. Around a contour of radius \( r \), the circulation is

\[
\Gamma = \int_0^{2\pi} u_\theta r \; d\theta = 2\pi C. \tag{3.32}
\]

This shows that \( \Gamma \) is constant, independent of the radius. (Compare this with the case of solid-body rotation, for which Eq. (3.29) shows that \( \Gamma \) is proportional to \( r^2 \).) In fact, the circulation around a circuit of any shape that encloses the origin is \( 2\pi C \).

Now consider the implication of Stokes' theorem

\[
\Gamma = \int_A \omega \cdot dA, \tag{3.31}
\]

for a contour enclosing the origin. The left-hand side of Eq. (3.31) is nonzero, which implies that \( \omega \) must be nonzero somewhere within the area enclosed by the contour. Because \( \Gamma \) in this flow is independent of \( r \), we can shrink the contour without altering the left-hand side of Eq. (3.31). In the limit the area approaches zero, so that the vorticity at the origin must be infinite in order that \( \omega \cdot \delta A \) may have a finite nonzero limit at the origin. We have therefore demonstrated that the flow represented by

\[
u_{\theta} = \frac{C}{r} \quad \text{is irrotational everywhere except at the origin, where the vorticity is infinite. Such a flow is called an irrotational or potential vortex.}
\]

Although the circulation around a circuit containing the origin in an irrotational vortex is nonzero, that around a circuit not containing the origin is zero. The circulation around any such contour \( ABCD \) (Figure 3.15) is

\[
\Gamma_{ABCD} = \int AB u \cdot ds + \int BC u \cdot ds + \int CD u \cdot ds + \int DA u \cdot ds.
\]

Because the line integrals of \( u \cdot ds \) around \( BC \) and \( DA \) are zero, we obtain

\[
\Gamma_{ABCD} = -u_\theta r \Delta \theta + (u_\theta + \Delta u_\theta)(r + \Delta r) \Delta \theta = 0,
\]

where we have noted that the line integral along \( AB \) is negative because \( u \) and \( ds \) are oppositely directed, and we have used \( u_\theta r = \text{const} \). A zero circulation around \( ABCD \) is expected because of Stokes' theorem, and the fact that vorticity vanishes everywhere within \( ABCD \).

**Rankine Vortex**

Real vortices, such as a bathtub vortex or an atmospheric cyclone, have a core that rotates nearly like a solid body and an approximately irrotational far field (Figure 3.16a). A rotational core must exist because the tangential velocity in an irrotational vortex has an infinite velocity jump at the origin. An idealization of such a behavior is called the Rankine vortex, in which the vorticity is assumed uniform within a core of radius \( R \) and zero outside the core (Figure 3.16b).
12. One-, Two-, and Three-Dimensional Flows

A truly one-dimensional flow is one in which all flow characteristics vary in one direction only. Few real flows are strictly one dimensional. Consider the flow in a conduit (Figure 3.17a). The flow characteristics here vary both along the direction of flow and over the cross section. However, for some purposes, the analysis can be simplified by assuming that the flow variables are uniform over the cross section (Figure 3.17b). Such a simplification is called a one-dimensional approximation, and is satisfactory if one is interested in the overall effects at a cross section.

A two-dimensional or plane flow is one in which the variation of flow characteristics occurs in two Cartesian directions only. The flow past a cylinder of arbitrary cross section and infinite length is an example of plane flow (Note that in this context the word "cylinder" is used for describing any body whose shape is invariant along the length of the body. It can have an arbitrary cross section. A cylinder with a circular cross section is a special case. Sometimes, however, the word "cylinder" is used to describe circular cylinders only.)

Around bodies of revolution, the flow variables are identical in planes containing the axis of the body. Using cylindrical polar coordinates \( (R, \varphi, z) \), with \( x \) along the axis of the body, only two coordinates \( (R \) and \( x \) \) are necessary to describe motion (see Figure 6.27). The flow could therefore be called "two dimensional" (although not plane), but it is customary to describe such motions as three-dimensional axisymmetric flows.

13. The Streamfunction

The description of incompressible two-dimensional flows can be considerably simplified by defining a function that satisfies the law of conservation of mass for such flows. Although the conservation laws are derived in the following chapter, a simple and alternative derivation of the mass conservation equation is given here. We proceed from the volumetric strain rate given Eq. (3.9), namely,

\[
\frac{1}{\rho} \frac{D \rho}{Dt} \left( \frac{\partial \mathbf{u}}{\partial t} \right) = \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial x_i}.
\]

The \( D/\rho \) signifies that a specific fluid particle is followed, so that the volume of a particle is inversely proportional to its density. Substituting \( \delta \gamma \propto \rho^{-1} \), we obtain

\[
\frac{1}{\rho} \frac{D \rho}{Dt} = \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial x_i},
\]

This is called the continuity equation because it assumes that the fluid flow has no voids in it; the name is somewhat misleading because all laws of continuum mechanics make this assumption.

The density of fluid particles does not change appreciably along the fluid path; under certain conditions, the most important of which is that the flow speed should be small compared with the speed of sound in the medium. This is called the Boussinesq approximation and is discussed in more detail in Chapter 4, Section 18. The condition holds in most flows of liquids, and in flows of gases in which the speeds are less than
about 100 m/s. In these flows $\rho^{-1} \, D \rho / D t$ is much less than any of the derivatives in $\partial u_i / \partial x_i$, under which condition the continuity equation (steady or unsteady) becomes

$$\frac{\partial u_i}{\partial x_i} = 0.$$  

In many cases the continuity equation consists of two terms only, say

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$  \hspace{1cm} (3.33)

This happens if $u$ is not a function of $z$. A plane flow with $u = 0$ is the most common example of such two-dimensional flows. If a function $\psi(x, y, t)$ is now defined such that

$$u = \frac{\partial \psi}{\partial y},$$
$$v = -\frac{\partial \psi}{\partial x},$$  \hspace{1cm} (3.34)

then Eq. (3.33) is automatically satisfied. Therefore, a streamfunction $\psi$ can be defined whenever Eq. (3.33) is valid. (A similar streamfunction can be defined for incompressible axisymmetric flows in which the continuity equation involves $R$ and $z$ coordinates only; for compressible flows a streamfunction can be defined if the motion is two dimensional and steady (Exercise 2.).

The streamlines of the flow are given by

$$\frac{dx}{u} = \frac{dy}{v},$$  \hspace{1cm} (3.35)

Substitution of Eq. (3.34) into Eq. (3.35) shows

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0,$$

which says that $d\psi = 0$ along a streamline. The instantaneous streamlines in a flow are therefore given by the curves $\psi = \text{const.}$, a different value of the constant giving a different streamline (Figure 3.18).

Consider an arbitrary line element $d\mathbf{x} = (dx, dy)$ in the flow of Figure 3.18. Here we have shown a case in which both $dx$ and $dy$ are positive. The volume rate of flow across such a line element is

$$vd\mathbf{x} + (-u) d\mathbf{y} = -\frac{\partial \psi}{\partial x} dx - \frac{\partial \psi}{\partial y} dy = -d\psi.$$

showing that the volume flow rate between a pair of streamlines is numerically equal to the difference in their $\psi$ values. The sign of $\psi$ is such that, facing the direction of motion, $\psi$ increases to the left. This can also be seen from the definition equation (3.34), according to which the derivative of $\psi$ in a certain direction gives the velocity component in a direction $90^\circ$ clockwise from the direction of differentiation. This requires that $\psi$ in Figure 3.18 must increase downward if the flow is from right to left.

One purpose of defining a streamfunction is to be able to plot streamlines. A more theoretical reason, however, is that it decreases the number of simultaneous equations to be solved. For example, it will be shown in Chapter 10 that the momentum and mass conservation equations for viscous flows near a planar solid boundary are given, respectively, by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial y^2},$$  \hspace{1cm} (3.36)
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$  \hspace{1cm} (3.37)

The pair of simultaneous equations in $u$ and $v$ can be combined into a single equation by defining a streamfunction, when the momentum equation (3.36) becomes

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^3 \psi}{\partial y^3} = 0.$$

We now have a single unknown function $\psi$ and a single differential equation. The continuity equation (3.37) has been satisfied automatically.

**Summarizing**, a streamfunction can be defined whenever the continuity equation consists of two terms. The flow can otherwise be completely general, for example, it can be rotational, viscous, and so on. The lines $\psi = C$ are the instantaneous streamlines, and the flow rate between two streamlines equals $d\psi$. This concept will be generalized following our derivation of mass conservation in Chapter 4, Section 3.
14. Polar Coordinates

It is sometimes easier to work with polar coordinates, especially in problems involving circular boundaries. In fact, we often select a coordinate system to conform to the shape of the body (boundary). It is customary to consult a reference source for expressions of various quantities in non-Cartesian coordinates, and this practice is perfectly satisfactory. However, it is good to know how an equation can be transformed from Cartesian into other coordinates. Here, we shall illustrate the procedure by transforming the Laplace equation

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},$$

to plane polar coordinates.

Cartesian and polar coordinates are related by

$$\begin{align*}
x &= r \cos \theta, & \theta &= \tan^{-1}(y/x), \\
y &= r \sin \theta, & r &= \sqrt{x^2 + y^2}.
\end{align*}$$

(3.38)

Let us first determine the polar velocity components in terms of the streamfunction. Because $\psi = f(x, y)$, and $x$ and $y$ are themselves functions of $r$ and $\theta$, the chain rule of partial differentiation gives

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial \theta} + \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial r} \frac{\partial r}{\partial \theta}.$$

Omitting parentheses and subscripts, we obtain

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial x} \cos \theta + \frac{\partial \psi}{\partial y} \sin \theta = -v \cos \theta + u \sin \theta.$$

(3.39)

Figure 3.19 shows that $u_\theta = v \cos \theta - u \sin \theta$, so that Eq. (3.39) implies $\partial \psi / \partial r = -u_\theta$. Similarly, we can show that $\partial \psi / \partial \theta = r u_r$. Therefore, the polar velocity components are related to the streamfunction by

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta},$$

$$u_\theta = \frac{\partial \psi}{\partial r}.$$

This is in agreement with our previous observation that the derivative of $\psi$ gives the velocity component in a direction $90^\circ$ clockwise from the direction of differentiation.

Now let us write the Laplace equation in polar coordinates. The chain rule gives

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial x} \frac{\partial x}{\partial \theta}.$$

Differentiating this with respect to $x$, and following a similar rule, we obtain

$$\frac{\partial^2 \psi}{\partial x^2} = \cos \theta \frac{\partial}{\partial r} \left[ \frac{\cos \theta \partial \psi}{\partial r} - \frac{\sin \theta \partial \psi}{\partial \theta} \right] - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left[ \frac{\cos \theta \partial \psi}{\partial r} - \frac{\sin \theta \partial \psi}{\partial \theta} \right].$$

(3.40)

In a similar manner,

$$\frac{\partial^2 \psi}{\partial y^2} = \sin \theta \frac{\partial}{\partial r} \left[ \frac{\sin \theta \partial \psi}{\partial r} + \frac{\cos \theta \partial \psi}{\partial \theta} \right] + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left[ \frac{\sin \theta \partial \psi}{\partial r} + \frac{\cos \theta \partial \psi}{\partial \theta} \right].$$

(3.41)

The addition of Eqs. (3.40) and (3.41) leads to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0,$$

which completes the transformation.

**Exercises**

1. A two-dimensional steady flow has velocity components

$$u = y, \quad v = x.$$ 

Show that the streamlines are rectangular hyperbolas

$$x^2 - y^2 = \text{const.}$$

Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a $90^\circ$ corner.

2. Consider a steady axisymmetric flow of a compressible fluid. The equation of continuity in cylindrical coordinates $(R, \varphi, z)$ is

$$\frac{\partial}{\partial R} (R u_R) + \frac{\partial}{\partial x} (R u_x) = 0.$$
Show how we can define a streamfunction so that the equation of continuity is satisfied automatically.

3. If a velocity field given by $u = ay$, compute the circulation around a circle of radius $r = 1$ about the origin. Check the result by using Stokes' theorem.

4. Consider a plane Couette flow of a viscous fluid confined between two flat plates at a distance $b$ apart (see Figure 9.4c). At steady state the velocity distribution is

$$u = Uy/b, \quad v = w = 0,$$

where the upper plate at $y = b$ is moving parallel to itself at speed $U$, and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

$$\psi = \frac{Uy^2}{b} + \text{const}.$$

5. Show that the vorticity for a plane flow on the $xy$-plane is given by

$$\omega_z = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right).$$

Using this expression, find the vorticity for the flow in Exercise 4.

6. The velocity components in an unsteady plane flow are given by

$$u = \frac{x}{1+t} \quad \text{and} \quad v = \frac{2y}{2+t}.$$

Describe the path lines and the streamlines. Note that path lines are found by following the motion of each particle, that is, by solving the differential equations

$$\frac{dx}{dt} = u(x, t) \quad \text{and} \quad \frac{dy}{dt} = v(x, t),$$

subject to $x = x_0$ at $t = 0$.

7. Determine an expression for $\psi$ for a Rankine vortex (Figure 3.16b), assuming that $u_0 = U$ at $r = R$.

8. Take a plane polar element of fluid of dimensions $dr$ and $d\theta$. Evaluate the right-hand side of Stokes' theorem

$$\int \omega \cdot dA = \int u \cdot ds,$$

and thereby show that the expression for vorticity in polar coordinates is

$$\omega_z = \frac{1}{r} \left[ \frac{\partial}{\partial \theta} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right].$$

Also, find the expressions for $\omega_r$ and $\omega_\theta$ in polar coordinates in a similar manner.

9. The velocity field of a certain flow is given by

$$u = 2xy^2 + 2xz^2, \quad v = x^2y, \quad w = x^2z.$$

Consider the fluid region inside a spherical volume $x^2 + y^2 + z^2 = a^2$. Verify the validity of Gauss' theorem

$$\int \nabla \cdot u \, dV = \int u \cdot dA,$$

by integrating over the sphere.

10. Show that the vorticity field for any flow satisfies

$$\nabla \cdot \omega = 0.$$

11. A flow field on the $xy$-plane has the velocity components

$$u = 3x + y \quad v = 2x - 3y.$$

Show that the circulation around the circle $(x - 1)^2 + (y - 6)^2 = 4$ is $4\pi$.

12. Consider the solid-body rotation

$$u_\theta = \omega r \quad u_r = 0.$$

Take a polar element of dimension $r \, d\theta$ and $dr$, and verify that the circulation is vorticity times area. (In Section 11 we performed such a verification for a circular element surrounding the origin.)

13. Using the indicial notation (and without using any vector identity) show that the acceleration of a fluid particle is given by

$$a = \frac{\partial u}{\partial t} + \nabla \left( \frac{1}{2} \dot{\theta}^2 \right) + \omega \times u,$$

where $\dot{\theta}$ is the magnitude of velocity $u$ and $\omega$ is the vorticity.

14. The definition of the streamfunction in vector notation is

$$u = -k \times \nabla \psi,$$

where $k$ is a unit vector perpendicular to the plane of flow. Verify that the vector definition is equivalent to Eqs. (3.34).

**Supplemental Reading**

