Problem Set # 2

1. Given an incompressible 2D flow with “x”” and “z” velocity given by

\[ u = \alpha x \]
\[ w = \beta z \]

(a) how are \( \alpha \) and \( \beta \) related?

(b) If the initial positions (t = 0) are given by X, Z what are the fluid positions x, z, in terms of X,Z, t?

(c) Sketch the x,z fluid element trajectory

(d) Express the velocity field in Lagrangian coordinates.

2 The following is a very simple model for coastal upwelling on a shelf of depth d and width L.

\[ u = \frac{x}{L} U \]
\[ w = -\frac{Wz}{d} \quad -d \leq z \leq 0 \]

where the ocean surface is at \( z = 0 \), \( x = 0 \) is taken as the “western” boundary. Note that \( z \) is negative downward.

(a) Take \( d = 100 \) meters, \( L = 50 \) km, \( W = 1 \) mm/sec. How long does it take a fluid element located on the bottom 10 km from the coast to reach a depth of 50 meters? What is its velocity at that point. How far is it from the coast at that point?

(b) How long does it take this fluid element to reach the surface? How far from the coast is the fluid point when it reaches the surface? Explain your answer and comment on what range of values of x and z this model should be valid.

3. For a 2d pure shear flow

\[ u = \alpha z \]
\[ w = 0 \]

(a) If the initial positions (t = 0) are given by X, Z what are the fluid positions x, z, in terms of X, Z, t?
(b) Consider the effect of this flow on the circle of fluid shown below, using \( d = 1 \) meter, \( \alpha = 0.02 \text{ sec}^{-1} \). Show where the circle of fluid shown below evolves after \( t = 60 \) seconds. How much has the circle of fluid rotated?

4. Suppose a fluid is characterized by the three components of velocity of

\[
\begin{align*}
    u &= 2xyz \\
    v &= -4yz^2 \\
    w &= 3z^2y
\end{align*}
\]

(a) Show that the fluid incompressible.
(b) What are the tree components of vorticity?
(c) What are the three components of the advection term in the Navier Stokes equation?

5. A certain flow field can be characterized as two dimensional. (Can you think of an example of this?) If the stream function is given by \( \psi = \frac{1}{2}(x^2 + y^2) \) (a) What are the \( u \) and \( v \) components of this flow? What is the vorticity of the flow? What direction is it in? What is the Lagrangian trajectory of the flow field. Describe the nature of the flow field.

6. A certain flow filed can be characterized given by \( \psi = \frac{1}{2}(x^2 - y^2) \) (a) What are the \( u \) and \( v \) components of this flow? What is the vorticity of this flow?
7. Oceanographers often do measurements between density surfaces, which are called layers. Let the density surfaces be identified by $\rho_1$, $\rho_2$. A measurement at a latitude of $30^\circ$ N shows a layer thickness of $H_1 = 100$ m with no motion within. The layer is traced to a latitude of $60^\circ$ N where it is observed that $H_2 = 200$. Use the concept of Conservation of Potential Vorticity to answer the following.

What is the total, planetary and relative vorticity at $30^\circ$ N? What is the total, planetary and relative vorticity at $60^\circ$ N?