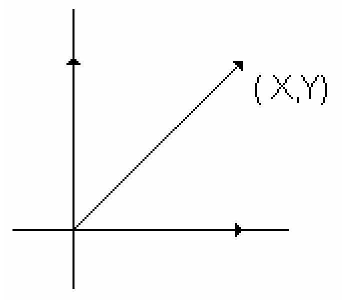
Math Review

We are already familiar with the concept of a scalar and vector.

Example: Position (x, y) in two dimensions



$s = (x^2 + y^2)^{\frac{1}{2}}$

where s is the length of

$\vec{x} = (x, y) = x\vec{i} + y\vec{j}$
And

$\vec{i}$, $\vec{j}$ are unit vectors in the “x” and “y” directions respectively with

$\vec{i} \bullet \vec{i} = 1$

$\vec{j} \bullet \vec{j} = 1$

$\vec{i} \bullet \vec{j} = 0$

We also write x_j , $j = 1, 2$
or

x_j = 1,2,3…

Matrices are “two component” objects with the same number of rows and columns

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn} \end{pmatrix}$$

$$A_{ij}\,(row, column)$$

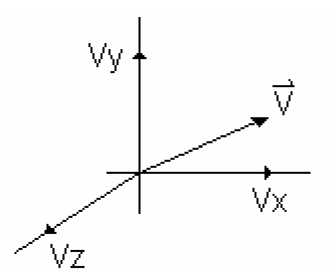
Concept of vector Magnitude

$\vec{V}$ is a Vector

$\left|\vec{V}\right|$ is the magnitude of a three dimensional vector

$$\left|\vec{V}\right| = (V_x^2 \;+ V_y^2 + V_z^2)^{\frac{1}{2}}$$

Note that sometimes the subscripts 1,2,3 are used for x,y,z.



Matrix Determinant

$$\left|A\right| = \det\left|\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right|$$

$$\left|A\right| = a_{11}(a_{22}a_{33} - a_{23}a_{23})$$

$$-a_{12}(a_{21}a_{33}-a_{23}a_{31})$$
$$+a_{31}(a_{12}a_{23}-a_{22}a_{13})$$

Dot product of two vectors

(n=3)

$$A_i \cdot B_i = \sum_{i=1}^{n} A_i \; B_i = A_1B_1 + A_2B_2 + A_3B_3$$
$$= |A||B|\cos(\theta)$$

Repeated indices indicate summation

Matrix Multiplication

$$C_{ij} = A \cdot B \; = A_{ik} \cdot B_{kj}$$

2D Example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} (a_{11}b_{11}+a_{12}b_{21}) & (a_{11}b_{12}+a_{12}b_{22}) \\ (a_{21}b_{11}+a_{22}b_{21}) & (a_{21}b_{12}+a_{22}b_{22}) \end{pmatrix}$$

Properties of the determinant of a Matrix

$$|A| = \det(A)$$

$$|A \cdot B| = |A| \cdot |B|$$

Trace

$$Tr(A) = \sum_{i=1}^{n} A_{ii}$$

No similar rule for trace

Matrix Inverse

$$(A^{-1})_{ij} A_{jk} = \delta_{jk}$$

Where δ_{jk} is the unity (sometimes called Kronecker delta) matrix with

$$\delta_{ij} = 1 \quad for \quad \mathrm{j = k}$$
$$= 0 \quad for \quad \mathrm{j \neq k}$$

$$\delta = \left\{ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\}$$

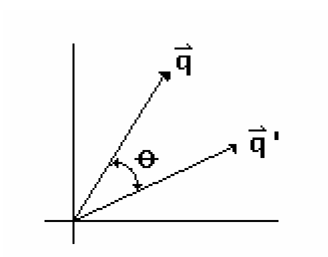
Another useful matrix operation is the transpose defined by

$$A^T{}_{ij} = A_{ji}$$

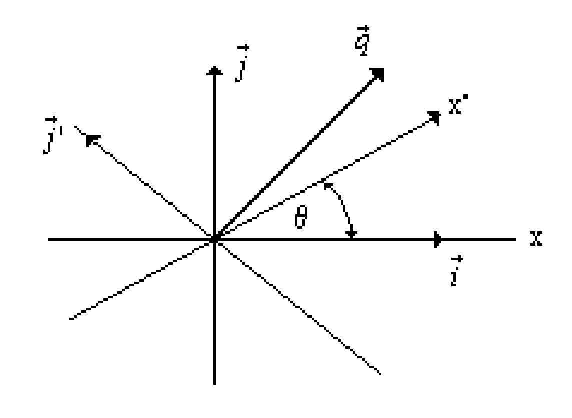
characterized by the rows and columns shifted.

Concept of Tensors

Objects such as scalars, vectors, matrices and others such as
B_{ijk} (a "third order" object re indices) can undergo transformations (not defined yet) if these objects remain as the same type we call these tensors.
The transformation that we will consider is rotation



We see that $|\vec{q}'| = |\vec{q}|$
We can also look at this problem a bit differently



We keep $\vec{q}$ fixed but rotate the axis. We say that we are rotating (or transforming from the old coordinate system x, y into the new coordinate system x',y')

We can write

$$\vec{q} = q_x\vec{i} + q_y\vec{j}$$

$$\vec{q} = q'_{x'}\vec{i}' + \vec{q}_{y'}\vec{j}'$$

where $\vec{i}$ and $\vec{j}$ are unit vectors

Note that $\vec{q}$ does not change in the coordinate transformation but rather "where it is" in the new coordinate system does change.

Let 1, 2 refer to x, y, i.e.

$$q_1 = q_x$$

$$q_2 = q_y$$

$$\vec{i} = (1,0)$$

$$\vec{j} = (0,1)$$

Note:

$$\vec{i} = (\vec{i}\cdot\vec{i}')\vec{i}' + (\vec{i}\cdot\vec{j}')\vec{j}'$$

$$\vec{j} = (\vec{j}\cdot\vec{i}')\vec{i}' + (\vec{j}\cdot\vec{j}')\vec{j}'$$

$$\vec{i} = \cos\alpha_{11}\vec{i}' + \cos\alpha_{12}\vec{j}'$$

$$\vec{j} = \cos\alpha_{21}\vec{i}' + \cos\alpha_{22}\vec{j}'$$

α_{ij} is the directional angle from the old to new axis and we have used our definition of the dot product.

Note:

$\vec{i}\cdot\vec{i}=1$, $\vec{j}\cdot\vec{j}=1$, $\vec{i}\,'\cdot\vec{i}\,'=1$, $\vec{j}\,'\cdot\vec{j}\,'=1$, $\vec{i}\cdot\vec{j}=0$ and $\vec{i}\,'\cdot\vec{j}\,'=0$ $\Rightarrow$

$$\cos^2\alpha_{11}+\cos^2\alpha_{12}=1$$
$$\cos^2\alpha_{21}+\cos^2\alpha_{22}=1$$

$$\cos\alpha_{11}=\cos\theta$$
$$\cos\alpha_{12}=-\sin\theta$$
$$\cos\alpha_{21}=\sin\theta$$
$$\cos\alpha_{22}=\cos\theta$$

Then:

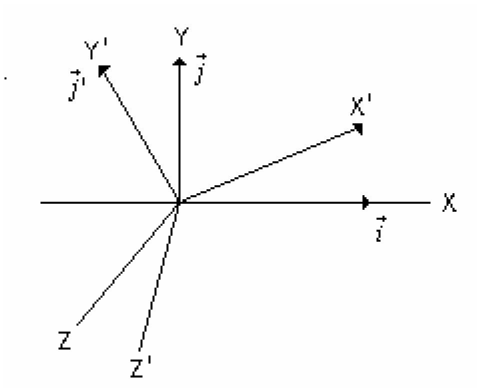
$$q_1'=\quad q_1\cos\theta+q_2\sin\theta$$
$$q_2'=-q_{x'}\sin\theta+q_{y'}\cos\theta$$
$$q_i=R_{ij}q_j$$

$$\left|R_{ij}\right|=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}q'_1\\ q'_2\end{pmatrix}$$

3D: Use directional cosines for rotation

$$R_{ij} = \cos\alpha_{ij}$$
$$i = 1,2,3$$
$$j = 1,2,3$$



$$\left|\vec{i}\right| = 1$$
$$\vec{i} = (\vec{i}\cdot\vec{i}\,')\vec{i}\,' + (\vec{i}\cdot\vec{j}\,')\vec{j}\,' + (\vec{i}\cdot\vec{k}\,')\vec{k}\,'$$
$$\vec{i} = \cos\alpha_{11}\vec{i}\,' + \cos\alpha_{12}\vec{j}\,' + \cos\alpha_{13}\vec{k}$$

etc.

Note

$$\sum_{j=i}^{3}\cos^2\alpha_{ij} = 1 \quad \text{for all i}$$

Properties of the Rotation Matrix R

$$R_{ij}^{\ T} = R_{ji} = (R^{-1})_{ij}$$
$$and$$
$$\det(\mathrm{R}) = 1$$

These properties of a matrix constitute “orthogonality” and the rotation matrix is an orthogonal matrix.

Note the definition of the inverse of a matrix

$$(R^{-1})_{ij}R_{jk} = \delta_{ik}$$

since

$$(R^{-1})_{ij} = R^{T}{}_{jk} = R_{kj}$$

$$R_{ij} R_{kj} = \delta_{ik}$$

Note: det (R) = 1 follows from the above equation and $\det(\delta) = 1$

Physical Application of scalars, vectors, 2nd order tensors:

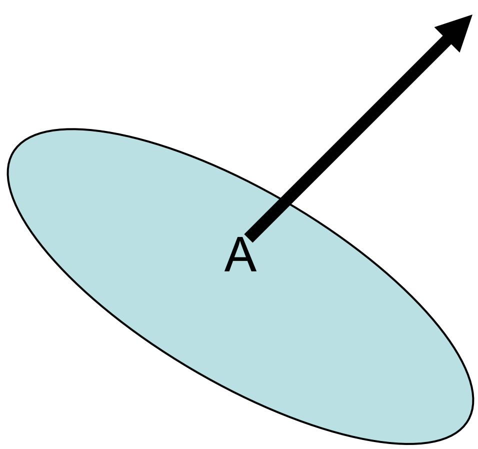
Scalars: Temperature, Pressure, Length, Speed

Vectors: Force, Velocity, and Acceleration

2nd Order Tensors: Stress tensor, Rate of Strain Tensor

Concept of Stress

Stress is a force per unit area

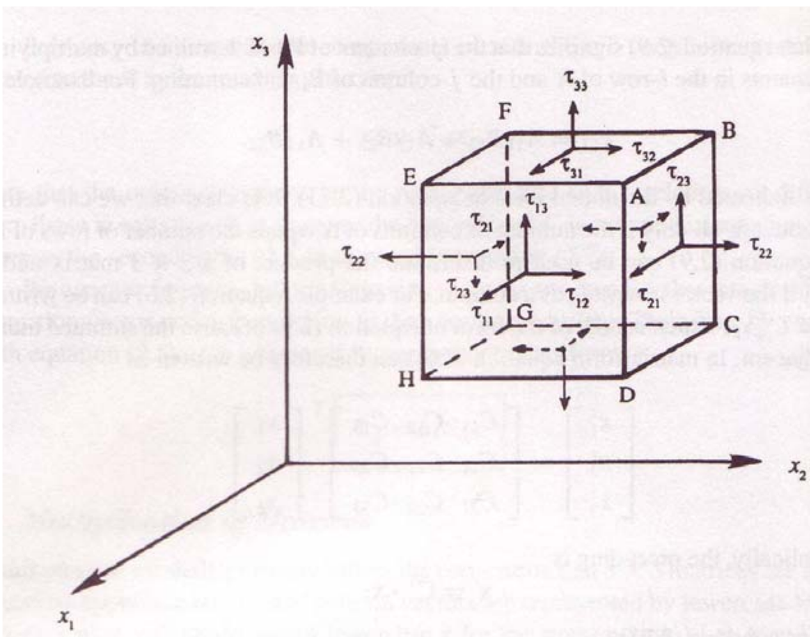


Given some area, A which is sufficiently small to be considered a plane surface. We define the vector $\vec{A}$ as the magnitude of the area $\vec{A}$ directed normal to the surface. $\vec{A}$ is now a vector! We sometimes write

$\vec{A} = A\vec{n}$

Where A is the magnitude of the area and $\vec{n}$ is a unit vector in the direction of $\vec{A}$

Stress is defined as force applied on a particular vector area. We will show that stress is a second order tensor. Shown below is a figure of Cohen and Kundu. Note the directions of the 1,2,3 directions. We define τ_{ij} as the force in the on area "i" in the "j" direction. See Figure below.



In three dimensions we can write the stress tensor as (in matrix form)

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Note that the force on an area $\vec{A}$ is given by

$$f_j = \tau_{ij} A_i$$

or

$$\vec{f} = \vec{\tau}_j A_j$$

Coordinate Transfer of second tensors (matrices)

Consider some 2nd order tensor B_{ij}'

$$B_{ij}' = A_{ie}^T B_{ek} A_{kj}$$

We can write B_{ij} as

$$B_{ij} = \frac{1}{2}\left[(B_{ij} + B_{ji}) + (B_{ij} - B_{ji})\right]$$

$$B_{ij} = C_{ij} + D_{ij}$$

Note that

$$C = C^T$$

and thus $C_{ij} = C_{ji}$ is symmetric
and $D_{ij} = -D_{ji}$ is antisymmetric

$$D = -D^T$$

Theorem: you can find an orthogonal transformation (rotation) R_{ij} such that it will "diagonalize" a symmetric 2nd order tensor

$$C_{ij}' = R_{ie}^T C_{ek} R_{kj} = \lambda_i \delta_{ij}$$

The quantities λ_i are called the eignenvalues. This equation can be interpreted as transforming C_{ij}

$$C_{ij} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Into λ_i are called Eigenvalues

$$C'_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

NOTE: $\det(C') = \det(R)\det(C)\det(R)$

$\det(C') = \lambda_1\lambda_2\lambda_3$

Since $\det(R) = 1$

$\det(C) = \lambda_1\lambda_2\lambda_3$

Also it can be shown that

$$R_{ni}R_{ei}C_{ek}R_{kj} = \lambda_i\delta_{ij}R_{ni}$$

~~λ_i (not summed)~~

results in

$$C_{nk}R_{kj} = \lambda_i\delta_{ij}R_{ni} = \lambda_j R_{nj}$$

Which is the eigenvalue and eigenvector Equations.

We can also write the above equation as

$$(C_{en} - \lambda_j\delta_{ln})R_{nj} = 0$$

resulting in

$$\det(C_{en} - \lambda_j \delta_{ln}) = 0$$

or

$$\det\begin{pmatrix} C_{11}-\lambda_1 & C_{12} & C_{13} \\ C_{12} & C_{22}-\lambda_2 & C_{23} \\ C_{13} & C_{23} & C_{33}-\lambda_3 \end{pmatrix} = 0$$

This equations determines the eigenvalues λ_i
Once your λ's are known, the eignefunctions can be calculated.

Example Rate of Strain tensor for a pure shear.

We will be examining in the next lecture the rate of strain tensor defined as

$$\mathrm{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Note that $\mathrm{E}_{ij} = \mathrm{E}_{ji}$ and E_{ij} is symmetric. For a pure shear, velocity is in one direction andits gradient is perpendicular to it. Consider a two dimensional flow u_i with

$$u_2 = 0$$
$$\textit{and}$$
$$\frac{du_1}{dx_2} = \alpha$$

$$E_{ij} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}$$

where $\Gamma = \dfrac{\alpha}{2}$

From above we obtain the eignenvalues from

$$\det\begin{pmatrix} \lambda & \Gamma \\ \Gamma & \lambda \end{pmatrix} = \lambda^2 - \Gamma^2 = 0$$

$$\Rightarrow \qquad \lambda = \pm\Gamma$$

yielding

$$\lambda_1 = \Gamma$$
$$\lambda_2 = -\Gamma$$

The eigenfunctions equations are

$$E_{nk} R_{kj} = \lambda_j R_{nj}$$

Or

$$\begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 R_{11} & \lambda_2 R_{12} \\ \lambda_1 R_{21} & \lambda_2 R_{22} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 R_{11} & \lambda_2 R_{12} \\ \lambda_1 R_{21} & \lambda_2 R_{22} \end{pmatrix}$$

$$row\#1 * column1$$
$$R_{21}\Gamma = \lambda_1 R_{11} = \Gamma R_{11} \Rightarrow R_{21} = R_{11}$$
$$row\#2 * column1$$
$$R_{11}\Gamma = \lambda_1 R_{21} = \Gamma R_{11} \Rightarrow R_{11} = R_{21}$$
$$row\#1 * column2$$
$$R_{22}\Gamma = \lambda_2 R_{12} = -\Gamma R_{12} \Rightarrow R_{22} = -R_{12}$$
$$row\#2 * column2$$
$$R_{12}\Gamma = \lambda_2 R_{21} = -\Gamma R_{22} \Rightarrow R_{12} = -R_{22}$$

Solving and requiring that det (R) = 1

Results in

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Which represents a rotation of 45^o !

Each column of R represent the two eignefunctions.

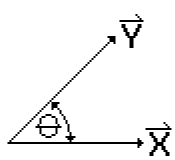
Math Operation

We have already discussed the "dot" product

$$X\vec{i} \cdot Y\vec{i}$$

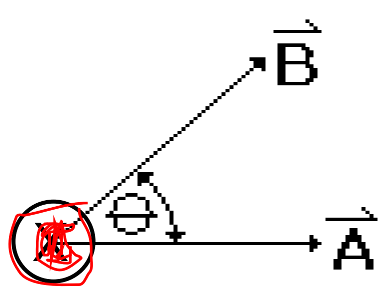
$$\vec{X} \cdot \vec{Y} = XY \cos\theta$$

$$\vec{X} \cdot \vec{Y} = \sum_{i=1}^{3} X_i Y_i$$



Note that the dot product of two vectors yields a scalar.
The Cross Product of two vectors yields another vector.

$$\vec{A} x \vec{B} = \vec{C}$$



$$\left|\vec{C}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta$$

$\vec{C}$ is directed perpendicular to the plane of A,B by the right –hand screw rule and points out of the paper, which is indicated by the "x".

Levi-Cevita Symbol $\in_{ijk}$ also called the Alternating Tensor. It is a third order tensor! One of its main usage is in defining the cross product.

$$\in_{ijk} = \begin{cases} 1 & \text{for i, j, k in sequence} \\ -1 & \text{for i, j, k not in sequence} \\ 0 & \text{if any two indices are repeated} \end{cases}$$

examples

$$\in_{123} = \in_{312} = \in_{231} = 1$$

$$\in_{321} = \in_{132} = \in_{213} = -1$$

An imp[ortant relationship (qwhich is also a homework problem is

$$\in_{ijk} \in_{klm} = \delta_{ie}\delta_{jm} - \delta_{im}\delta_{je}$$

One can write

$$\vec{C} = \vec{A}x\vec{B}$$

$$C_i = \in_{ijk} A_j B_k$$

Vector Calculus

We want to generalize some one-dimensional calculus ideas to 2 and 3 dimensions

F=F(x)

$$F = F\left(x\right) = \int \frac{dF}{dx}dx$$

If F(x) =0, x$\rightarrow$-∞

$$F\left(x\right) = \int_{-\infty}^{x} \frac{dF}{dx}dx$$

Sometimes a function has more than one variable

F(x, y, z, t)

Which we can write as

$$F\left(\vec{x}, t\right)$$

Consider:

$$\frac{\partial F}{\partial x_i}$$

Where
$$\begin{aligned} i &= 1,2,3 \\ x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned}$$

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x}$$

Implied is y,z, are held constant.

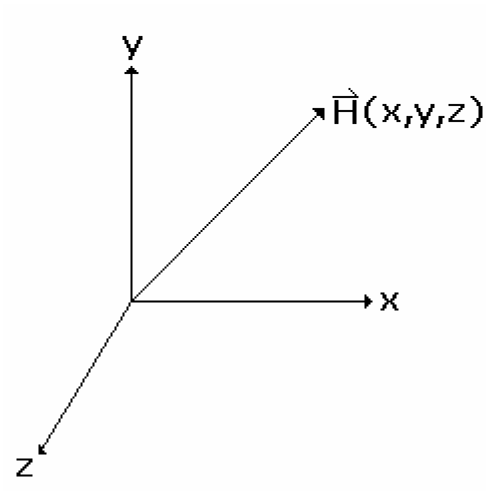
$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x} = \lim_{\delta x \to 0} \frac{F(x+\delta x, y, z) - F(x, y, z)}{\delta x}$$

$$G_i = \frac{\partial F}{\partial x_i}$$

$$\vec{G} = \nabla F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial x}\vec{j} + \frac{\partial F}{\partial x}\vec{k}$$

Curl of a vector

$\vec{\nabla} x \vec{H}$



$\vec{\nabla} x \vec{H}$,take derivation in direction perpendicular to H and then form a vector component perpendicular to both of them/

$$\vec{C} = \nabla x \vec{H} = \in_{ijk} \frac{\partial H_k}{\partial x_j}$$

Consider some function
$\phi = \phi(x, y, z)$

The equation $\phi = \phi(x, y, z)$=C
C is a constant defines a surface in three dimensions.

Solve for say z = F[x,y;C]

We can write

$$\nabla \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial x}\vec{j} + \frac{\partial \phi}{\partial x}\vec{k}$$

However if $\phi(x, y, z)$ were constant then

$\nabla \phi = 0$
Thus

$\nabla\phi$ is perpendicular to the tangent to the surface ϕ = Constant

Gauss' Law

In one dimension fro F = F(x)

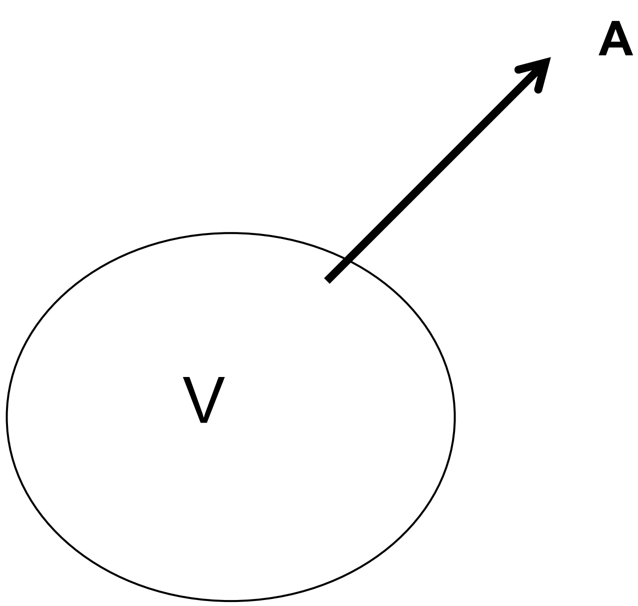
$$\int_a^Y \frac{dF}{dx}dx = F(Y) - F(a)$$

Consider now the three dimensional situation

$$\int \frac{\partial F}{\partial x_i} d^3x = \int F dA_i$$

where the area dA_i is normal to the volume and represents its surface. An important subcase is Gauss' Law

$$\int \nabla \cdot \vec{F} d^3x = \int \vec{F} \cdot d\vec{A}$$



Stokes Theorem

$$\int \left(\vec{\nabla} \times \vec{u}\right) \cdot dA = \int \vec{u} \cdot d\vec{s}$$

Around loop