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Application of the Robinson biodynamical theory to turbulence

Louis Goodman *

School for Marine Science and Technology, University of Massachusetts Dartmouth, New Bedford, MA 02744, United States

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ABSTRACT

In the late 1990s Allan Robinson developed a theory of NPZ interactions in a laminar upwelling flow field. His approach was to use the advection reaction (AR) equation in a Lagrangian coordinate system. Recently, his theory was extended to turbulent flow by applying a probability density function to the solution of the AR equation. A review of this work is presented as well as new work examining the role of the turbulent induced biodynamical interaction (TIBI) effect, which is typically neglected in advection diffusion reaction (ADR) formulations for NPZ problems. The TIBI effect is associated with turbulence inducing fluctuations in nonlinear biological constituent interactions and is separate from the effect of turbulent mixing in dispersing the constituents. A simple example of the application of the theory – that of nutrient and phytoplankton fields being upwelled into a uniform optically active turbulent mixed layer – is presented. For this example, not including the TIBI term in an ADR formulation results in an overestimate of the primary production, increasing with decreasing turbulent Peclet number.

1. Introduction

In the late 1990s Allan Robinson developed a novel approach to modeling the interaction of nutrients, N, phytoplankton, P, and zooplankton, Z (Robinson, 1997, 1999). His approach was to model NPZ interactions by the advection reaction (AR) equation and utilize a Lagrangian coordinate sys-

* Tel.: +1 508 9106375.
E-mail address: lgoodman@umassd.edu

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tem to obtain solutions. With this methodology, he examined a number of example NPZ interactions undergoing advection into an optically active upper layer. However, the upper ocean is typically characterized by turbulent rather than purely laminar flow. To accomplish the goal of including turbulence, an extension of the basic Robinson AR theory was required.

There are two general approaches to turbulent NPZ problems. These are: (1) embedding a NPZ interaction model in a numerical simulation of the turbulent field; or (2) embedding a NPZ interaction model in the advection diffusion reaction (ADR) equation. Examples of approach (1) include: Yamazaki and Kamykowski (1991) and Visser (1997), both of whom use random walk simulations to model NPZ interactions in an optically active turbulent mixed layer. Three significant issues arise with this approach. First, it is computationally intensive and becomes very limited in the type of turbulence model and mean background flow fields which can be used. Care must be applied in properly taking into account advective effects (Ross and Sharples, 2004). Second, and more subtle, the boundary conditions prescribed are either perfectly reflecting or perfectly absorbing and do not capture the nature of a balance between advection and diffusion at the mixed layer lower boundary. More realistic boundary conditions which prescribe some fraction of the material allowed to escape the mixed layer would require a further empirical assumption.

The second approach is that of using the NPZ interaction model as the "reactive" term of the advective diffusive reaction (ADR) equation. See, for example, Wroblewski (1977) and Franks (2002). That approach neglects the effect of the Turbulence Induced Biodynamical Interactions (TIBI) which arise from the inherent non linearity of the biodynamical interactions between N, P, Z. Donaghay and Osborn (1997) have recognized the importance of the TIBI term and suggested using a simple linearization of the interaction term. Note that the TIBI effect should be distinguished from that of turbulent mixing, the latter typically modeled by some type of eddy diffusivity.

Allan Robinson and I collaborated on extending his biodynamical theory to include turbulence. The approach, following that of Pope (1994), was to apply a probability density function (PDF) for the turbulent displacement field to the starting point of the Robinson biodynamical theory, namely the AR equation. This collaboration culminated in a manuscript (Goodman and Robinson, 2008) applying a PDF model based on a random walk to the Robinson biodynamical theory. We reworked a number of his original examples and established that the resulting phytoplankton growth was a strong function of the turbulent Peclet number, the ratio of the advective to the turbulent time scale. Most recently, and just prior to his untimely death, Allan and I developed a theory to obtain a turbulent PDF as a solution to the AD equation. This allowed a direct comparison of the effect of including, or not including, the TIBI term in the ADR equation approach to the NPZ interaction problem.

In this manuscript I will review the basic biodynamical theory set forth by Allan Robinson (Section 2), and our initial work in its extension to turbulence (Section 3). These sections are a review of the theory described in manuscripts Robinson (1997, 1999) and in Goodman and Robinson (2008). Sections 4 and 5 contain new work on which Allen and I recently collaborated. Section 4 presents a prescription for obtaining a turbulent PDF compatible with including the TIBI term in the ADR equation, while Section 5 applies these results to a simple NP interaction problem. Section 6 contains the summary and conclusions.

2. Robinson biodynamical theory

The starting point typically used in NPZ interaction models is the advection reaction (AR) equation (Robinson, 1997; Franks, 2002)

$$\frac{d\phi_i}{dt} = F_i[\phi_1, \phi_2, \ldots \phi_n; \vec{x}, t], \tag{1}$$

where ϕ_i, $i = 1, \ldots n$, are any one of n biological state variables, i.e. N, P, Z, for $n = 3$, and F_i is the reaction (growth/decay rate) term describing the biodynamical interaction between the state variables. In general F_i has explicit space, $\vec{x}$, and time, t, dependence due to external factors such as the light field. See Franks (2002) for a review of models of F_i.

The left hand term of (1) is recognized as the total time derivative, defined by

$$\frac{d\phi_i}{dt} \equiv \left(\frac{\partial \phi_i}{\partial t}\right)_{\vec{X},t_0} = \left(\frac{\partial \phi_i}{\partial t}\right)_{\vec{x}} + \vec{u}\cdot\nabla(\phi_i), \tag{2}$$

which can also expressed in terms of the Lagrangian coordinates $\vec{X}, t, t_0$ through $\vec{x}=\vec{x}(\vec{X},t,t_0)$, the trajectory of a fluid element at time, t, whose initial position is $\vec{X}$, at time, t_0. The velocity field is then given by $\vec{u}=(\partial\vec{x}/\partial t)_{\vec{X},t,t_0}$.

In a series of manuscripts, Robinson (1997, 1999) showed that by using a Lagrangian coordinate system, solutions of Eq. (1) can be obtained. To see this consider, the ratio

$$\frac{d\phi_i/dt}{d\phi_j/dt} = \frac{d\phi_i}{d\phi_j} = \frac{F_i[\phi_1,\phi_2,\ldots,\phi_n;\vec{x},t]}{F_j[\phi_1,\phi_2,\ldots,\phi_n;\vec{x},t]} = \frac{F_i[\phi_1,\phi_2,\ldots,\phi_n;\vec{X},t]}{F_j[\phi_1,\phi_2,\ldots,\phi_n;\vec{X},t]},$$

which for $i \neq j$ leads to $n-1$ set of independent equations and solutions of the form

$$\phi_j = \psi_j(\phi_i,\vec{X},t)|_{j\neq i}. \tag{3}$$

In general, the solutions given by (3) are not in closed form, i.e. the functional dependence on $\vec{X}$, t are, in general, unknown. Substitution of (3) into (1) yields a set of n uncoupled equations of the form

$$\frac{d\phi_i}{dt} = R_i[\phi_i;\vec{X},t,t_0], \tag{4}$$

where R_i is the reaction term. For application to the NPZ upwelling problem, the constraints imposed on the solution to (4) are:

Constraint A, $\phi_i(\vec{X},t,t_o)$ specified at $t=t_o$ over some boundary $\vec{X}$; and

Constraint B, the Lagrangian trajectory $\vec{x}(\vec{X},t,t_o)$ specified for $t \geq t_o$, over $\vec{X}$. Using (4), Robinson (1999) examined a number of cases of laminar upwelling for various model optically active NPZ interactions.

3. Extension to turbulence

To include turbulence, the fluid velocity is decomposed into a mean and fluctuating component,

$$\vec{u} = \vec{U} + \vec{u}', \tag{5}$$

where $\vec{U}=\langle\vec{u}\rangle$. To proceed further, some additional assumption on the nature of the random field fluctuations has to be made. Typically, in turbulence applications to scalar mixing, an eddy diffusivity model is assumed, whence the constituent particle flux term is given by $\langle\phi_i'\vec{u}_i'\rangle = -\overleftrightarrow{k}\cdot\nabla\bar{\phi}_i$, where the eddy diffusivity, $\overleftrightarrow{k}$, is represented in its most general tensor form to allow for different directional mixing and spatial dependence. With this assumption, ensemble averaging Eq. (1), and using (2) and (3), we obtain an equation for the evolution of the mean field of $\bar{\phi}_i$, namely

$$\left\langle\frac{d\phi_i}{dt}\right\rangle = \frac{\partial\bar{\phi}_i}{\partial t} + \vec{U}\cdot\nabla\bar{\phi}_i - \nabla\cdot(\overleftrightarrow{k}\cdot\nabla\bar{\phi}_i) = \langle F_i[\phi_j;\vec{x},t]\rangle, \tag{6}$$

where the overbar is used interchangeably with the angular brackets to indicate an ensemble average. At this point this equation cannot be solved unless the RHS term, $\langle F_i[\phi_j,\vec{x},t]\rangle$, is determined or known in terms of the moments of ϕ_j. However, it should be noted that it is not Eq. (6) that is used as the starting point in NPZ modeling, but rather

$$\frac{\partial\bar{\phi}_i}{\partial t} + \vec{U}_i\cdot\nabla\bar{\phi}_i - \nabla\cdot(\overleftrightarrow{k}\cdot\nabla\bar{\phi}_i) = F_i[\bar{\phi}_j;\vec{x},t], \tag{7}$$

where, implicitly, the assumption is made that

$$\langle F_i[\phi_j;\vec{x},t]\rangle = F_i[\bar{\phi}_j;\vec{x},t]. \tag{8}$$

See, for example, Wroblewski (1977) and Franks (2002). However, note that, in general,

$$F_i[\phi_j] = F_i[\bar{\phi}_j + \phi_j'] = F_i[\bar{\phi}_j] + F_i', \tag{9}$$

where $\langle F_i' \rangle$ is the turbulence induced biodynamical interaction (TIBI) term. In Eq. (9) the notation for the space and time dependence is suppressed but space and time dependence are still, in general, explicitly included. The TIBI term, $\langle F_i' \rangle$, represents the effect of turbulence on the non-linear coupling within $F_i[\phi_j]$.

Rather than solve the ADR equation (6), Goodman and Robinson (2008) applied a model PDF to the solution of the AR equation (4) to obtain solutions for the mean biodynamical variables and for the mean interaction term, $\langle F_i \rangle$. They used a PDF associated with a random walk undergoing perfect reflection at the top and bottom of the mixed layer. In following the approach of Robinson (1997, 1999), the solution to the AR equation is obtained in terms of Lagrangian coordinates. To obtain statistics of the biodynamical state variables at fixed locations, the PDF required must be expressed in terms of Eulerian coordinates. This was accomplished by using Bayes' theorem, Eq. (2.11) of Goodman and Robinson (2008).

For the simple bilinear interaction case, they found that there are environmental parameter regimes in which the TIBI term was of order the mean term, $|\langle F_i' \rangle| \sim |F_i[\bar{\phi}_j]|$. The key parameter in determining the effect of the turbulent field on the TIBI term was found to be the turbulent Peclet number, defined as the ratio of turbulent diffusion to advective time scale.

However, the model PDF used in Goodman and Robinson (2008) does not satisfy the AD equation and the mean value of the biological constituents, i.e. $\bar{N}, \bar{P}, \bar{Z}$, do not satisfy the ADR equation. To be able to examine the effect of the TIBI term in the ADR equation, a PDF has to be obtained which does satisfy the AD equation and which does result in the mean biological constituents satisfying the ADR equation. We will now show that, by using the boundary condition of a balance between the advective and turbulent flux at the lower boundary of the mixed layer and no advective and no turbulent flux at the upper boundary, a PDF can be derived which does satisfy the AR equation with the mean biodynamical variables, i.e. $\bar{N}, \bar{P}, \bar{Z}$, satisfying the ADR equation.

4. A PDF based on the AD equation

Note that the homogenous, non-reactive version of (6), i.e. the LHS, is the advection diffusion (AD) equation, whose solution, $\tilde{\rho}$, satisfies

$$\frac{\partial \tilde{\rho}}{\partial t} + \vec{U}_i \cdot \nabla \tilde{\rho} - \nabla \cdot (\overset{\leftrightarrow}{k} \cdot \nabla \tilde{\rho}) = 0. \tag{10}$$

Using the normalization

$$\int d\vec{x} \tilde{\rho} = 1,$$

$\tilde{\rho} = \tilde{\rho}(\vec{x}; \vec{X}, t, t_0)$ can be interpreted as the conditional probability density function (PDF) associated with a random walk displacement, $\vec{x}$, undergoing advection by velocity, $\vec{U}_i$ (Papoulis, 1965). This PDF describes the statistics of the fluid particle displacement, $\vec{x}$, conditioned on its present time, t, its initial location, $\vec{X}$, and its initial time, t_0. At this point no boundary condition has been imposed on the PDF. Consider the conditional PDF associated with the initial time, t_0, as the independent random variable but conditioned on $\vec{x}, t; \vec{X}$. That is, given a particle at location $\vec{x}$ at time t, t_0 is the time at which the biological constituent particle was located at $\vec{X}$, which will be taken as the position of the boundary where the biological constituent particles enter the turbulent domain. This conditional PDF is then given by $\rho(t_o; \vec{x}, \vec{X}, t) = C\tilde{\rho}$; C is a normalizing constant defined by

$$\int dt_0 \rho(t_o; \vec{x}, \vec{X}, t) = 1, \tag{11}$$

where ρ, like $\tilde{\rho}$, satisfies the AD equation (10). Consider the mean quantity

$$\bar{\phi}_i = \int dt_0 \phi_i(\vec{X}, t, t_o)\rho(t_o; \vec{x}, \vec{X}, t). \tag{12}$$

We will now show that $\bar{\phi}_\iota$ satisfies an ADR equation with the reaction term containing the TIBI term. This follows since

$$\frac{d\bar{\phi}_\iota}{dt} = \left(\frac{d\bar{\phi}_\iota}{dt}\right)_{\vec{X},t,t_0} = \int dt_0 \left\{ \left(\frac{d\phi_\iota}{dt}\right)_{\vec{X},t,t_0} \rho(t_o; \vec{x}, \vec{X}, t) + \phi_\iota \left(\frac{d\rho}{dt}\right)_{\vec{X},t,t_0} \right\}$$
$$= \int dt_0 \{G(\phi_\iota)\rho(t_o; \vec{x}, \vec{X}, t) + \phi_\iota \nabla \cdot (\overleftrightarrow{k} \cdot \nabla\rho)\}.$$

Since $\phi_\iota = \phi_\iota(\vec{X}, t, t_0)$, it then follows that

$$\int dt_0\ \phi_\iota \nabla \cdot (\overleftrightarrow{k} \cdot \nabla\rho)\} = \nabla \cdot (\overleftrightarrow{k} \cdot \nabla\bar{\phi}_i),$$

which results in

$$\frac{d\bar{\phi}_\iota}{dt} - \nabla \cdot (\overleftrightarrow{k} \cdot \nabla\bar{\phi}_\iota) = \left\langle F(\phi_\iota)\right\rangle. \tag{13}$$

To illustrate an application of this result and an examination of the role of the TIBI term in the ADR equation (13), a simple example – upwelling of seed nutrients and phytoplankton into a turbulent optically active mixed layer – is presented.

5. Example: TIBI effect for a bilinear NP interaction

Consider the simple example used in previous manuscripts by Robinson (1999) and Goodman and Robinson (2008), namely, that of a linear strain upwelling of seed nutrients and phytoplankton into an optically active turbulent mixed layer. The equations for this model are

$$\frac{d\tilde{P}}{dt} = \left(\frac{\partial\tilde{P}}{\partial t}\right)_z + \tilde{w}\left(\frac{\partial\tilde{P}}{\partial z}\right)_t = \beta\tilde{N}\tilde{P} \tag{14a}$$

$$\frac{d\tilde{N}}{dt} = \left(\frac{\partial\tilde{N}}{\partial t}\right)_z + \tilde{w}\left(\frac{\partial\tilde{N}}{\partial z}\right)_t = -\beta\tilde{N}\tilde{P}, \tag{14b}$$

where $\tilde{P}$, $\tilde{N}$ are the phytoplankton, nutrient linear densities (units of m^{-3}), respectively; β is the nutrient uptake rate; $\tilde{w} = -\alpha z$ is the vertical velocity with a constant strain rate, $-\alpha$; z is taken as positive downward. The turbulent mixed layer is located between $z = 0$ and $z = D$. The mean flow field is two dimensional and incompressible. It is assumed that one dimensional turbulent mixing dominates, i.e. turbulent scales in the horizontal are much larger than that in the vertical. Adding (14a) and (14b) result in the total biomass density being conserved in a Lagrangian coordinate system, i.e.

$$M_0 = \tilde{N} + \tilde{P}, \tag{15}$$

where $M_0 = N_0 + P_0$ is the biomass density at $t = t_0$. Normalizing the variables in equations (14a) and (14b) using the mixed layer depth D, total biomass density, M_0, and the nutrient uptake time $\tau = 1/\beta M_0$ yields

$$\frac{\tilde{P}}{M_0} \to P; \quad \frac{\tilde{N}}{M_0} \to N; \quad P + N = 1; \quad t\tau \to t; \quad \frac{\alpha}{\tau} \to \alpha; \quad \frac{z}{D} \to z$$

and results in the normalized set of equations

$$\frac{dP}{dt} = \left(\frac{\partial P}{\partial t}\right)_z - \alpha z\left(\frac{\partial P}{\partial z}\right)_t = PN \tag{16a}$$

$$\frac{dN}{dt} = \left(\frac{\partial N}{\partial t}\right)_z - \alpha z \left(\frac{\partial N}{\partial z}\right)_t = -PN, \tag{16b}$$

which have the solutions

$$P = P(Z, t, t_0) = \frac{P_0}{P_0 + N_0 \exp[-(t - t_0)]}, \tag{17a}$$

$$N = 1 - P, \tag{17b}$$

with P_0, N_0 the normalized phytoplankton and nutrient seed densities, respectively, at $t = t_0$. To obtain the PDF our starting point is the advection diffusion equation for a linear strain rate mean flow and a constant vertical eddy diffusivity, κ. Eq. (10) for this case reduces to

$$\frac{\partial \hat{\rho}}{\partial t} - \alpha z \frac{\partial \hat{\rho}}{\partial z} - \kappa \frac{\partial^2 \hat{\rho}}{\partial z^2} = 0. \tag{18}$$

For boundary conditions we assume that the total flux of material (the sum of the advective and turbulent components) vanish at the base of the mixed layer and both the advective and turbulent flux vanish at the top of the mixed layer, whence

$$\alpha z \hat{\rho} + \kappa \frac{\partial \hat{\rho}}{\partial z} = \kappa \frac{\partial \hat{\rho}}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{19a}$$

$$\alpha z \hat{\rho} + \kappa \frac{\partial \hat{\rho}}{\partial z} = \alpha z \rho_0 \quad \text{at} \quad z = 1. \tag{19b}$$

where ρ_0 is the density of material entering the mixed layer Eq. (18) with boundary conditions (19a) and (19b) describes the evolution of a scalar non-interacting density field, $\hat{\rho}$, entering the mixed layer with initial density, ρ_0. To see the latter, consider a solution to (18) of the form

$$\hat{\rho} = \int_0^\infty d\tilde{\gamma} K(\tilde{\gamma}) \exp(-\tilde{\gamma}(t - t_0) + \rho_0,$$

where ρ_0 is the density at $z = 1$, $t = t_0$ and which, upon substitution into (18), yields

$$\tilde{\gamma} K + \alpha z \frac{\partial K}{\partial z} + \kappa \frac{\partial^2 K}{\partial z^2} = 0, \tag{20}$$

with the boundary conditions (19) becoming

$$\frac{\partial K}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{21a}$$

$$\alpha K + \kappa \frac{\partial K}{\partial z} = 0 \quad \text{at} \quad z = 1. \tag{21b}$$

Let

$$K = G(z) \exp\left(-\frac{Pe}{4} z^2\right),$$

where the turbulent Peclet number is given by

$$Pe = \frac{\alpha}{\kappa}.$$

where α is the non dimensional strain rate and κ the non dimensional turbulent diffusivity. Substitution of K into Eq. (20) yields

$$Pe \left[\gamma - \frac{1}{2} - Pe \frac{z^2}{4}\right] G + \frac{\partial^2 G}{\partial z^2} = 0, \tag{22}$$

where $\tilde{\gamma} = \alpha\gamma$. With the above substitution, G satisfies the boundary conditions

$$\frac{\partial G}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{23a}$$

$$\frac{Pe}{2}G + \frac{\partial G}{\partial z} = 0 \quad \text{at} \quad z = 1. \tag{23b}$$

It is straightforward to show that Eq. (22) with boundary conditions (23) result in eigenfunction solutions $G = G_m$, with associated eigenvalues, $\gamma = \gamma_m$, $m = 1, 2, \ldots, n, \ldots$ Note that G_m is orthonormal,

$$\int_0^1 dz\, G_m G_n = \delta_{mn}.$$

It is also straightforward to show that

$$G_1 = \frac{1}{c_1}\exp\left(-\frac{Pe}{4}z^2\right),$$
$$c_1^2 = \int_0^1 dz (G_1)^2,$$
$$\gamma_1 = 1.$$

Using these eigenfunctions and eigenvalues yields the solution

$$\hat{\rho} = \rho_0\left[1 - \sum_{m=1}^{m=\infty} A_m G_m c_1 G_1 \exp(-\alpha\gamma_m(t - t_0))\right], \tag{24}$$

where

$$A_m = \int_0^1 dz' G_m(z')\exp\left(\frac{Pe}{4}z'^2\right).$$

Eq. (24) describes the temporal and spatial evolution of the density field, $\hat{\rho}$, which is given by $\hat{\rho} = \rho_0$ at $t = t_0$ and $z = 1$. Note that when $t = \infty$, $\hat{\rho} = \rho_0$, as expected.

To obtain the PDF, we rewrite (24) as

$$\hat{\rho} = \int_{-\infty}^{t} dt' H(t_0, t')\left(-\frac{\partial\hat{\rho}}{\partial t'}\right) = \int_{-\infty}^{t} dt' \{\rho_0 H(t_0, t')\}\left\{\frac{1}{\rho_0}\left(-\frac{\partial\hat{\rho}}{\partial t'}\right)\right\}, \tag{25}$$

with the Heaviside function H defined by

$$H(t_0, t') = \begin{bmatrix} 1 & t' \geq t_0 \\ 0 & t' < t_0 \end{bmatrix}.$$

Eq. (25) can then be interpreted as the prescription for obtaining the average density $\hat{\rho} = \bar{\rho}$ from the initial density $\rho = \rho_0 H(t_0, t')$ using the PDF, $\hat{F} = \frac{1}{\rho_0}(-\partial\hat{\rho}/\partial t')$.Changing the independent random variable t' to $\tilde{z}$ according to $\tilde{z} = \exp[-\alpha(t - t')]$, results in

$$\bar{\rho} = \int_0^1 d\tilde{z}\, \rho(\tilde{z})\frac{\hat{F}(\tilde{z})}{d\tilde{z}/dt'} = \int_0^1 d\tilde{z}\, \rho(\tilde{z})F(\tilde{z}), \tag{26a}$$

$$F = \sum_{m=1}^{m=\infty} \gamma_m A_m G_m c_1 G_1 \tilde{z}^{(\gamma_m - 1)} = \frac{\partial Q}{\partial\tilde{z}} \tag{26b}$$

with

$$Q = \sum_{m=1}^{m=\infty} A_m G_m c_1 G_1 \tilde{z}^{\gamma_m}.$$

Note that F, as a PDF, has the proper normalization

$$\int_{-\infty}^{t} dt' F(t') = \int_{0}^{1} d\tilde{z} F(\tilde{z}) = \int_{0}^{1} d\tilde{z} \frac{\partial Q}{\partial \tilde{z}} = Q(1) - Q(0) = 1.$$

We can interpret $\tilde{z}$ as the Lagrangian position of a fluid element at time t', whose initial position at $z = 1$ was t. Ensemble averages of P and PN are then given by

$$\langle P \rangle = \int_{-\infty}^{t} dt' P(t') F(t') = \int_{0}^{1} d\tilde{z}\, P(\tilde{z}) F(\tilde{z}), \tag{27a}$$

$$\langle PN \rangle = \int_{-\infty}^{t} dt' P(t') N(t') F(t') = \int_{0}^{1} d\tilde{z}\, P(\tilde{z}) N(\tilde{z}) F(\tilde{z}). \tag{27b}$$

Using Eq. (26b) with (27a) and (27b), it is straightforward to show that the mean phytoplankton density $\bar{P}$ satisfies the ADR equation

$$\overset{(I)}{\frac{\partial \bar{P}}{\partial t}} - \alpha z \overset{(II)}{\frac{\partial \bar{P}}{\partial z}} - \kappa \overset{(III)}{\frac{\partial^2 \bar{P}}{\partial z^2}} = \overset{(IV)}{R}, \tag{28}$$

with boundary conditions

$$\frac{\partial \bar{P}}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{29a}$$

$$Pe\bar{P} + \frac{\partial \bar{P}}{\partial z} = PeP_0 \quad \text{at} \quad z = 0. \tag{29b}$$

The "reaction" term, R, is given by

$$R = \langle PN \rangle = R_0 + R_T;$$

with $R_0 = \bar{P}\bar{N}$ and $R_T = \langle P'N' \rangle$, the latter the TIBI term. Thus, we have formally solved the ADR equation (28) without recourse to dropping the TIBI term.

To examine the role of the TIBI term, R_T, we will compare the results out lined above to that of obtaining a solution to the ADR equation (28) with no TIBI term, i.e. $R = R_0 = \bar{P}\bar{N}$. To obtain numerical results we use as input parameters typical oceanic values of

$$\tilde{\alpha} \sim 10^{-6}\text{–}10^{-5} \quad \text{s}^{-1},$$
$$\tilde{\kappa} \sim 10^{-4}\text{–}10^{-2} \quad \text{m}^2 \quad \text{s}^{-1},$$
$$D \sim 10\text{–}50 \quad \text{m}$$

(Large et al., 1994; Goodman and Robinson, 2008). This yields a range of turbulent Peclet numbers of order $.01 < Pe < 10$. Note that Pe is independent of the nutrient uptake time, τ. If we use one day as a characteristic nutrient uptake time and the smaller value of the linear strain rate of $\tilde{\alpha} \sim 10^{-6}$, this results in the normalized strain rate $\alpha \sim .1$ (Robinson, 1999).

In Fig. 1 vertical profiles of $\bar{P}$ are shown for the steady state ($t \to \infty$, $\partial \bar{P}/\partial t = 0$) for $0 \le Pe \le \infty$ and $\alpha = .1$. The solid color coded lines are Case A, the solution for $\langle P \rangle$ using Eqs. (26b) and (27a), TIBI term included. The dashed color coded lines are solutions to the ADR equation (28) with the TIBI term neglected, $R_T = 0$, Case B, using the same range of Pe and $\alpha = .1$. The red solid line of Fig. 1 corresponds to no turbulence, pure advection, $Pe = \infty$. We see a very significant difference in the limiting value of $Pe = 0$ for the two cases. Fig. 1 shows that neglecting the TIBI term over a wide range of Pe, up to $Pe = 10$, where advection dominates turbulence, results in a large over estimate of $\bar{P}$. Compare the dashed with the solid lines. Thus, turbulence in this simple model tends to limit phytoplankton growth over a very wide range of Peclet numbers.

To understand this result we present in Fig. 2(a–d), for the steady state cases presented in Fig. 1, the contribution of the three steady state terms of Eq. (27) – advection, term II, blue lines; turbulent

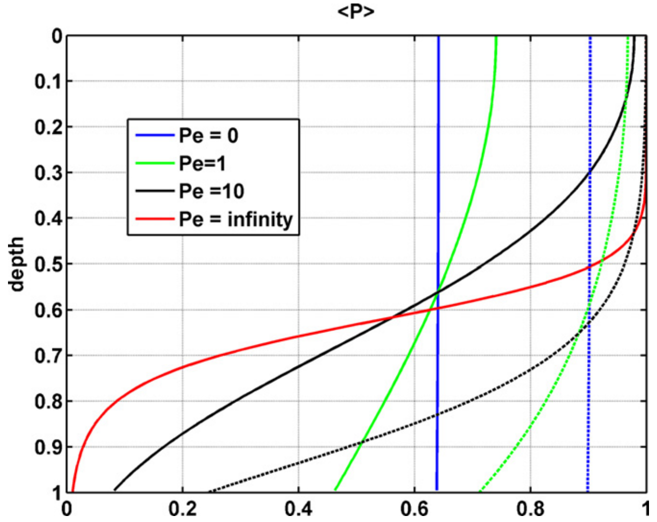

Fig. 1. Steady state ($t \to \infty$, $\partial\bar{P}/\partial t = 0$) vertical profiles of $\langle P\rangle$ for the Peclet numbers indicated in the figure, using $\alpha = .1$. Case A, the solid color coded lines are the solutions for $\langle P\rangle$ obtained by using (27a) with (26b) as the PDF, TIBI term included. Case B, the dashed color coded lines, the solution to the ADR equation (28) with the TIBI term neglected, $R = R_0$.

diffusion, term III, green lines; and growth rate (reaction), R, term IV, the black lines. As in Fig. 1, the solid lines refer to Case A, the dashed lines, Case B.

Note that the advection term II, blue line, is always negative and indicates an upward flux of material. The turbulent diffusion term, III, green line, can be of either sign, with negative indicating a downward flux of material, and positive, an upward flux. The growth rate term, III, black line, is always positive. The three terms must balance in the steady state.

In Fig. 2d, the no turbulence, pure advection (red line) regime, where $R_T = 0$, advection balances growth rate. For the turbulence dominated regime of $Pe = 0$, Fig. 2a, turbulent diffusion balances growth rate for both case A, including the TIBI term, and Case B, no TIBI term. In Figs. 1 and 2a, where $Pe = 0$, note the constant vertical distribution of $\bar{P}$ and constant vertical distribution of terms III and IV of the ADR equation (28). This results from vertically uniform mixing in the steady state. Turbulence strongly dominates advection, which is reflected in the vertically uniform PDF, $F = 1$. However, this result only depends on the relative value of the turbulent time scale to advective time scale as given by the definition of the turbulent Peclet number,

$$Pe = \frac{(\tilde{\kappa}/D^2)^{-1}}{\tilde{\alpha}^{-1}} = \frac{\tilde{\alpha}D^2}{\tilde{\kappa}},$$

and not on the absolute intensity of the turbulent field, i.e. value of $\tilde{\kappa}$ alone. We also see in Fig. 2a and b for turbulent Peclet numbers $Pe = 0.1$ that for Case A, TIBI term included, (solid green line), at all depths, turbulent diffusion fluxes phytoplankton toward the base of the mixed layer. This downward flux of phytoplankton interacts with the incoming upward moving seed nutrients and the growth rate (black lines). However at $Pe = 10$ there is a change in sign of turbulent diffusion with depth at $z \approx .65$. This results in an upward turbulent flux of phytoplankton for $z \gtrsim .65$. The upward flux diminishes the growth rate in that depth range, as indicated in Fig. 2c. Compare in Fig. 2c the black solid line to the black dashed line, the former being with TIBI, the latter no TIBI. Thus, the TIBI term affects growth rate through its feed back with the turbulent diffusion term. The advection term is also affected when the TIBI term is included. Relative to not including TIBI, it remains negative, but upwardly transporting less material when the diffusion term III is positive, and more material when the diffusion term is negative. These effects are also present for smaller values of Pe, where turbulence dominates, and result in the

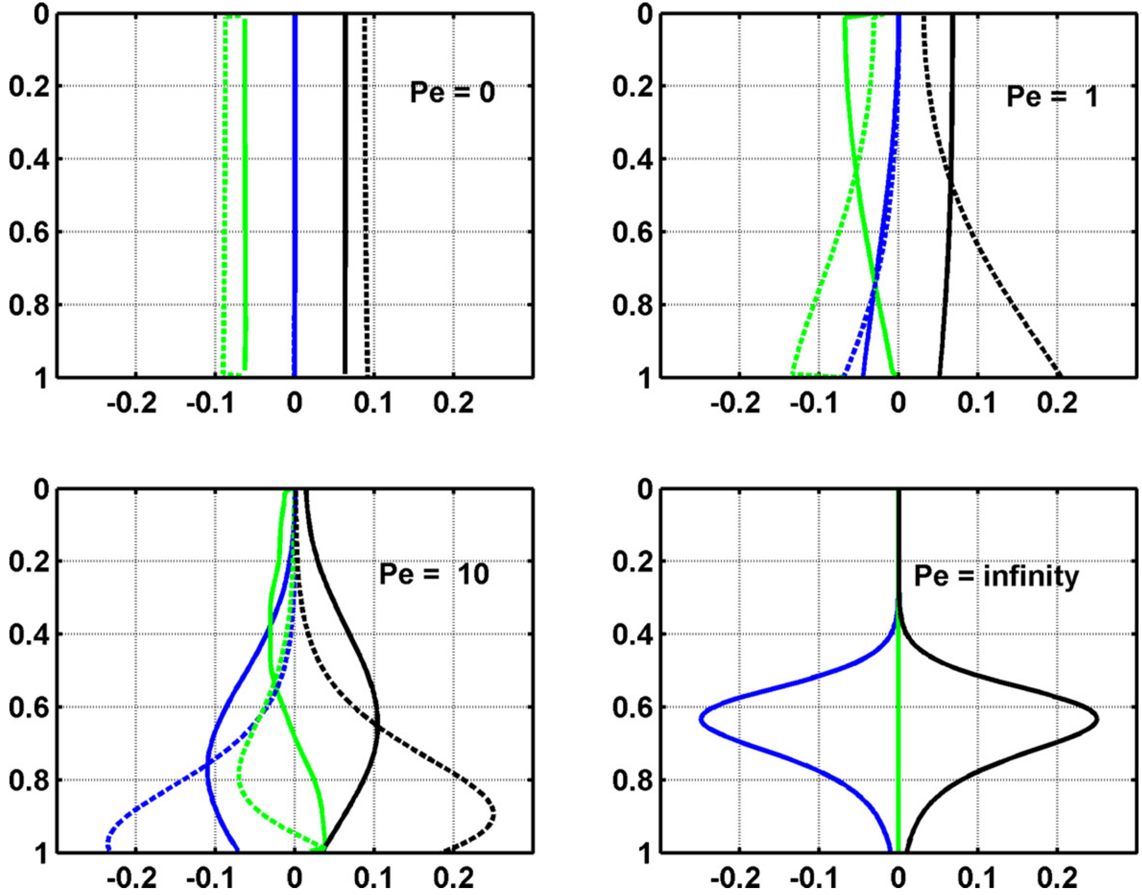

Fig. 2. Steady state ($t \to \infty$, $\partial\bar{P}/\partial t = 0$) contribution of the terms of equation (28) – advection, term II, blue lines; turbulent diffusion, term III, green lines; and the growth rate (reaction), R, term IV, black lines. As in Fig. 1, the solid lines refer to Case A, the dashed lines, Case B. *Note*: the advection term II, blue line, is always negative and indicates an upward flux of material; the turbulent diffusion term, III, green line, can be either sign with negative a downward flux of material and positive an upward flux; the reaction (growth rate) term, III, black line, is always positive. The three terms must balance in the steady state. (a) $Pe = 0$, (b) $Pe = 1$, (c) $Pe = 10$, (d) $Pe = \infty$.

different total integrated phytoplankton growth in the TIBI versus non-TIBI case. See Fig. 1, blue and green solid and dashed lines, respectively.

6. Summary and conclusions

In the late 1990s Allan Robinson developed a theory of NPZ interaction in a laminar upwelling flow field. His approach was to use the advection reaction equation (AR) and obtain solutions for the evolution of N, P, Z in a Lagrangian coordinate system (Robinson, 1997, 1999). The Robinson theory was extended to turbulent flow by Goodman and Robinson (2008) by using a probability density function (PDF) on the solution to the AR equation. The PDF employed was associated with a random walk undergoing perfect reflection at the top and bottom of the mixed layer. Bayes' theorem was used to express the PDF in Eulerian coordinates.

A simple bilinear NP turbulent upwelling interaction case was examined with this approach. It was shown that the key non dimensional parameter describing the evolution of the primary production was the turbulent Peclet number, the ratio of the advective to turbulent time scale. It was also observed that the Turbulence Induced Biodynamical Interaction (TIBI) term could not, in general, be neglected. The TIBI term arises from the effect of turbulence on the non-linear part of biodynamical interaction and is distinct from that of turbulent mixing.

However, the PDF used by Goodman and Robinson (2008) does not satisfy the AR equation nor does the resulting $\bar{N}, \bar{P}, \bar{Z}$ satisfy the advective diffusion reaction (ADR) equation. This resulted in an inability of that approach to be used to examine the role of the TIBI in the commonly used ADR models of NPZ interaction.

Allan and I, just prior to his untimely death, developed an approach, given in Section 4, to obtain a PDF which does satisfy the AD equation and also results in the mean biodynamical state variables $\bar{N}, \bar{P}, \bar{Z}$ satisfying the ADR equation. In Section 5, this approach is applied to the bilinear NP interaction example considered in previous manuscripts (Robinson, 1999; Goodman and Robinson, 2008), with particular emphasis on the role of the TIBI term. Except for extremely high Peclet numbers, $Pe \gg 10$, which correspond to advection effects dominating that of turbulence, the TIBI term results in a very significant contribution to the mean phytoplankton profile $\bar{P}$. Neglect of the TIBI term, as seen in Fig. 1, results in an overestimate of $\bar{P}$ with the overestimate increasing with decreasing Pe. Not including the TIBI term in the ADR equation also greatly alters the overall role that turbulence plays in determining the mean phytoplankton profile and in contributing to total phytoplankton production. As indicated in Fig. 1, for the limiting case of $Pe = \infty$, not including TIBI term, results in the mean total phytoplankton production in the steady state, $P_T = \int_0^1 dz\bar{P}$, being overestimated by approximately 40%. Also, as shown in Figs. 1 and 2, the vertical distribution of $\bar{P}$ and the three terms of the steady state ADR equation (27) are altered in the vertical over the intermediate range of Peclet numbers $1 < Pe < 10$. In particular the turbulent diffusion term (green lines, solid, no turbulence case, dashed lines, turbulent cases) show a sign reversal at depth for $1 < Pe < 10$.

The original theoretical framework developed by Allan Robinson lead to a PDF approach to modeling the TIBI term and most recently to a PDF which satisfies the AR equation and with $\bar{N}, \bar{P}, \bar{Z}$ satisfying the ADR equation. This allows a quantification of the role of TIBI in the ADR equation approach and a prescription on how to proceed with more complicated and realistic NPZ models in turbulent flows.

Acknowledgements

I dedicate this manuscript to my friend and colleague, Allan R. Robinson, who quite unexpectedly passed away while we working on it. He has been a great inspiration for interdisciplinary and novel techniques of modern oceanography and will be greatly missed. We also acknowledge support from ONR grant N000140810631 and program managers Dr. Jim Eckman, Dr. Scott Harper, and Dr. Terri Paluszkiewicz.

References

Donaghay, P.L., Osborn, T.R., 1997. Toward a theory of biological–physical control of harmful algal bloom dynamics and impact. Limnol. Oeanogr. 42 (5), 1283–1296.
Franks, P., 2002. NPZ models of plankton dynamic: their construction, coupling to physics, and application. J. Oceanogr. 58, 379–387.
Goodman, L., Robinson, A.R., 2008. On the theory of advective effects on biological dynamics in the sea. III. The role of turbulence in biological–physical interactions. Proc. R. Soc. Lond. A 464, 555–572.
Large, W.G., McWilliams, J.C., Doney, S.C., 1994. Oceanic vertical mixing: a review and a model with a non-local boundary layer parameterization. Rev. Geophys. 32, 363–403.
Papoulis, A., 1965. Probability, Random Variables and Stochastic Processes. McGraw-Hill, NY.
Pope, S.B., 1994. Lagrangian PDF methods for turbulent flows. Ann. Rev. Fluid Mech. 26, 23–63.
Robinson, A.R., 1997. On the theory of Advective effects on biological dynamics in the sea. Proc. R. Soc. Lond. A 453, 2295–2324.
Robinson, A.R., 1999. On the theory of Advective effects on biological dynamics in the sea, II Localization, Light limitation, and nutrient saturation. Proc. R. Soc. Lond. A 455, 1813–1828.
Ross, O.N., Sharples, J., 2004. Recipe for 1D Lagrangian particle tracking models in space varying diffusivity. Limnol. Oceanogr. 2, 289–302.
Visser, A.W., 1997. Using random walk models to simulate the vertical distribution of particles in a turbulent water column. Mar. Ecol. Progr. Ser. 158, 275–281.
Wroblewski, 1977. A model of phytoplankton plume formation during Oregon upwelling. JMR 35, 356.
Yamazaki, H., Kamykowski, 1991. The vertical trajectories of motile phytoplanktoin in a wind mixed water column. Deep-Sea Res. 38, 219–241.