TIHARC

June 3, 2005

1 PURPOSE:

TIHARC computes the harmonic constants for a particular station for which up to one year (8760 terms) of hourly sea level, bottom pressure, or current data is supplied. This routine is a modified version of a program which uses standard Fourier analysis plus traditional methods of the National Ocean Survey (as described in “A user’s Guide to a Computer Program for Harmonic Analysis of Data at Tidal Frequencies” by R.E. Dennis and E.E. Long in NOAA Technical Report NOS 41) for tidal computations. TIHARC differs in that the user is able to perform the analysis on any length series up to one year. Also included is an option to perform an error analysis on the results, which entails computing predicted and residual series from the observed data. For a discussion of this aspect of the program, see Appendix C.

The main analysis has three basic processes:

1. Initial separation of the tidal constituents from the data.
2. Orientation of the constituent tides with the astronomical elements.
3. Eliminating each desired constituent from the effects due to other constituents.

The equation

\[ h = H_o + \Sigma_n f_n H_n \cos \alpha_n t - (\kappa_n - [V_o + u]_n) \]  \hspace{1cm} (1)

describes the height of tide at any time for which

- \( H_o \) = mean value of tide for observations
- \( H_o = \frac{1}{N} \Sigma_{i=0}^{N-1} h_i \)
- \( t \) = time reckoned from an arbitrary reference
\( f_n = \text{node factor of constituent} \)

\( a_n = \text{constituent speed} \)

\( \kappa_n = \text{local epoch of the constituent at } t = 0 \text{ for period of observation} \)

\( (V_0 + u)_n = \text{value of argument of constituent at } t = 0 \text{ for the period of observation} \)

\( N = \text{total number of data values used in the analysis. The factor } H_n \text{ is the mean amplitude of the constituent, } n, \text{ for the entire nodal period whereas the adjustment using the node factor } f_n \text{, denoted} \)

\[
R_n = f_n H_n \tag{2}
\]

is the amplitude pertaining to a particular time. The amplitudes \( R_n \) are derived from observational data with Fourier techniques.

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Figure 1. Line spectra of tidal constituents at diurnal and semidiurnal frequencies. The Doodson numbers are shown for each constituent.
Although the constant $\kappa_n$ is usually used to characterize the constituent, the quantity

$$\zeta_n = \kappa_n - ((V_0 + u)_n)$$  \hspace{1cm} (3)

is actually calculated from the data, where $\kappa$ is the phase of the constituent at $t = 0$ for a particular set of observations. Phase is expressed here also as Greenwich epoch, $G$, which is defined as the phase relative to the Greenwich median and Greenwich time. $G$ and $\kappa$ are related as follows

$$G_n = \kappa_n + \text{west longitude } x \text{ species}$$  \hspace{1cm} (4)

See appendix A for a more thorough discussion of the phase relationships among terms used commonly in tidal analysis.

**Scalar Series Computation** In performing the harmonic analysis the user can specify the direct computation of either (a) all 25 constituents listed in table 1 or (b) just the first 10 constituents in which case the latter 15 constituents are inferred using equilibrium tidal considerations. Normally the direct computation of all 25 constituents is performed only on a “long” series so that the closely spaced tidal lines (see figure 1) can be adequately resolved. More specifically a series length in excess of 60d or 1440h can be considered “long”, although this definition is subjective and a more detailed discussion of this point appears in appendix B. In the case of a “long” series analysis (for which no inference is specified) the series is convolved with a Parzen window in order to reduce the end effects in the spectral domain. Normally, the direct computation of only the first 10 constituents with the inference of the last 15 constituents is performed on a “short” series with a length less than 60d or 1440h. In the case of a “short” series analysis (for which inference is specified) the series is not windowed.
Table 1: Tidal constituents computed by TIHARC. For the “short series” analysis the harmonic constants for the constituents indicated with * are inferred from using equilibrium tidal considerations. The equilibrium tidal coefficients are given relative to 0.536m.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Constituent (cph)</th>
<th>Frequency (cpd)</th>
<th>Equilibrium Coefficients</th>
<th>a(0/hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>0.0805114006</td>
<td>1.93227361</td>
<td>0.90812</td>
<td>28.98396</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0.0789992487</td>
<td>1.89598197</td>
<td>0.17387</td>
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</tr>
<tr>
<td>$S_2$</td>
<td>0.0833333333</td>
<td>2.00000000</td>
<td>0.42358</td>
<td></td>
</tr>
<tr>
<td>$K_1$</td>
<td>0.0417807462</td>
<td>1.00273791</td>
<td>-0.53050</td>
<td></td>
</tr>
<tr>
<td>$O_1$</td>
<td>0.0387306544</td>
<td>0.92953571</td>
<td>0.37689</td>
<td></td>
</tr>
<tr>
<td>$M_4$</td>
<td>0.1610228012</td>
<td>3.86454723</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$M_6$</td>
<td>0.2415342018</td>
<td>5.796820843</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$M_8$</td>
<td>0.3220456024</td>
<td>7.729094458</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$S_4$</td>
<td>0.1666666666</td>
<td>4.000000000</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$S_6$</td>
<td>0.2500000000</td>
<td>6.000000000</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$K_2*$</td>
<td>0.0835614924</td>
<td>2.00547582</td>
<td>0.11506</td>
<td></td>
</tr>
<tr>
<td>$L_2$</td>
<td>0.0820235526</td>
<td>1.96856526</td>
<td>-0.02567</td>
<td></td>
</tr>
<tr>
<td>$2N_2*$</td>
<td>0.0774870968</td>
<td>1.85960320</td>
<td>0.02301</td>
<td></td>
</tr>
<tr>
<td>$R_3*$</td>
<td>0.0834474075</td>
<td>2.00273778</td>
<td>-0.00354</td>
<td></td>
</tr>
<tr>
<td>$T_2*$</td>
<td>0.0832192592</td>
<td>1.99726222</td>
<td>0.02479</td>
<td></td>
</tr>
<tr>
<td>$\lambda_2*$</td>
<td>0.0818211814</td>
<td>1.963708354</td>
<td>-0.00670</td>
<td></td>
</tr>
<tr>
<td>$\mu_2*$</td>
<td>0.0776894679</td>
<td>1.86454723</td>
<td>0.02777</td>
<td></td>
</tr>
<tr>
<td>$\nu_2*$</td>
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<td>1.90083887</td>
<td>0.03303</td>
<td></td>
</tr>
<tr>
<td>$J_1*$</td>
<td>0.0432928982</td>
<td>1.03929555</td>
<td>-0.02964</td>
<td></td>
</tr>
<tr>
<td>$M_3*$</td>
<td>0.040265943</td>
<td>0.96644626</td>
<td>-0.0</td>
<td></td>
</tr>
<tr>
<td>$O01*$</td>
<td>0.0448308380</td>
<td>1.07594011</td>
<td>-0.01623</td>
<td></td>
</tr>
<tr>
<td>$P_1*$</td>
<td>0.0415525871</td>
<td>0.99726209</td>
<td>0.17554</td>
<td></td>
</tr>
<tr>
<td>$Q1*$</td>
<td>0.0372185025</td>
<td>0.89324406</td>
<td>0.07216</td>
<td></td>
</tr>
<tr>
<td>$2Q1*$</td>
<td>0.0357063506</td>
<td>0.85695241</td>
<td>0.00955</td>
<td></td>
</tr>
<tr>
<td>$\rho_1*$</td>
<td>0.0374208737</td>
<td>0.89810997</td>
<td>0.01371</td>
<td></td>
</tr>
</tbody>
</table>

A Fourier transform is performed on the data at specified tidal frequencies to determine the Fourier coefficients, which are used to compute (a) the constituent amplitude, $R_m$, for the period of the particular observations and (b) the phase, for the $n^{th}$ constituent at $t = 0$ of the observations. The values of amplitude, $H_0$, local epoch, $\kappa_n$ and Greenwich epoch, $G_n$, are found by correcting for long period tidal effects and adjusting phases in accordance with the astronomy. (Refer to Dennis and Long for a discussion of this step.)

For the user-specified "long series" computation, a lower-upper decomposition
method (see ref. Wendroff and Lax) is used in the elimination of constituent effects. For the user-specified "short series" computation, an overrelaxation matrix inversion method (see ref ...) is used in the elimination of constituent effects. A summary of the analysis is output to either the terminal or the line printer according to user preference. In addition, a constituent file, comprised of amplitude and local epoch for up to 25 harmonics and compatible with the tide prediction routine TIDHAR, can be output at the request of the user.

The analysis summary lists the variance for (a) the entire input data series, (b) the combined predicted diurnal constituents, (c) the combined predicted semidiurnal constituents and (d) the residual or observed minus the total predicted series. A comparison of these terms provides a crude measure of the quality of the analysis. The confidence limits for a particular constituent amplitude and phase can be determined using methods described in the attached appendix C.

Vector Series Computation:

A merged series of current components (northward (V) followed by eastward (U)) can be input into the tide analysis program. The program computes the harmonic constituents for each component using the methods described above. In addition to summary results for both the north and east currents, the ellipse characteristics including major and minor axes amplitudes, ellipse phase and orientation are computed for each constituent and are output. The convention here is to measure orientation clockwise from true north (direction of first component) and phase of the maximum current.

Figure 2. Tidal ellipse definitions

V - major axis
rV - minor axis
\( \tau \) - phase of maximum velocity V
\( \theta_1 \) - orientation of major axis relative to y axis
If the northward, \( V \), and eastward, \( U \), velocity components of a particular tidal constituent can be expressed in terms of their amplitudes, \( H_v \) and \( H_u \), and Greenwich phases, \( G_v \) and \( G_u \), according to

\[
V = H_v \cos(\omega t - G_v) = A_2 \cos \omega t - B_2 \sin \omega t \\
U = H_u \cos(\omega t - G_u) = A_1 \cos \omega t - B_1 \sin \omega t 
\]

then

\[
A_1 = H_u \cos G_u \quad A_2 = H_v \cos G_v \\
B_1 = H_u \sin G_u \quad B_2 = H_v \sin G_v .
\]

Using the convention according to Butman (1975), the major axis amplitude is

\[
V = \frac{1}{2} \left[ (A_1 + B_2)^2 + (A_2 - B_1)^2 \right]^{\frac{1}{2}} + \left[ (A_1 - B_2)^2 + (A_2 + B_1)^2 \right]^{\frac{1}{2}}
\]

and the minor axis amplitude is

\[
rV = \frac{1}{2} \left[ (A_1 + B_2)^2 + (A_2 - B_1)^2 \right]^{\frac{1}{2}} - \left[ (A_1 - B_2)^2 + (A_2 + B_1)^2 \right]^{\frac{1}{2}}.
\]

The phase of the maximum amplitude, \( \tau \), is given by

\[
\tau = \frac{1}{2} \tan^{-1} \left( \frac{-2(A_1 B_1 + A_2 B_2)}{A_1^2 - B_1^2 + A_2^2 - B_2^2} \right)
\]

and the orientation relative to north (+ \( \theta \) clockwise) is

\[
\theta = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \left( \frac{-2(A_1 A_2 + B_1 B_2)}{A_1^2 + B_1^2 - (A_2^2 + B_2^2)} \right)
\]

With these conventions, a positive (negative) minor axis value means the constituent tidal current rotates counterclockwise (clockwise).

2 EXECUTION:

OK, TIHARC

TIHARC: (DATE) (TIME)
LAST UPDATE: (DATE)
“A CURRENT SERIES CONSISTS OF MERGED NORTH, EAST COMPS”

“NAME OF INPUT FILE:” -enter filename.
(header is displayed)

“OUTPUT DEVICE:”
“ENTER ‘T’ TO OUTPUT TO YOUR TTY OR ‘L’ TO OUTPUT TO THE LPT”
-enter T or L.

“DO YOU WANT AN ERROR ANALYSIS?”
“ANSWER Y OR N” -enter Y or N

“DO YOU ANALYZE CURRENTS?”
“ANSWER Y OR N” -enter Y or N

If output choice was to TTY, then the following is displayed at the TTY, otherwise it goes to the line printer.

“START TIME”
(GREGORIAN DATE) (JULIAN DATE)

“END TIME”
(GREGORIAN DATE) (JULIAN DATE)

“REF. TIME MERIDIAN, STATION WEST LONG. -enter time meridian (Greenwich ≡ 0.0) and west long. (in degrees & fractions of degrees)

“TITLE” -enter a title you wish to appear above the analysis summary.

“DO YOU WISH TO INFER THE MINOR CONSTITUENTS”
“ANSWER Y OR N” -enter Y or N.
Note: For currents the following is repeated for both components, “North” component first.

“OUTPUT JUST THE RESULTS OF THE ANALYSIS”
“ANSWER Y OR N” -enter Y or N.
If Y, a summary table is output, if N, other diagnostic info. plus summary table output.

If the output choice was TTY, then the tables will be displayed at the user’s terminal, otherwise they will be sent to the line printer.

If an error analysis is performed then,
“DO YOU WANT TO OUTPUT PREDICTED SERIES, Y/N?”
-enter Y or N.
If Y, “NAME OF PREDICTED SERIES” -enter filename.
“DO YOU WANT TO OUTPUT A RESIDUAL SERIES Y/N?” -enter Y or N.
If Y: “NAME OF RESIDUAL SERIES” -enter filename
If the answer to “ERROR ANALYSIS” was Y, then option to output predicted, residuals series is printed.

“DO YOU WANT TO OUTPUT PREDICTED SERIES?”
“ANSWER Y OR N” -enter Y or N

If Y:
“ENTER SERIES NAME:” -enter filename

“DO YOU WANT TO OUTPUT RESIDUAL SERIES?”
“ANSWER Y OR N” -enter Y or N

If Y:
“ENTER SERIES NAME” -enter filename

“DO YOU WISH TO CONSTRUCT A CONSTITUENT FILE?”
“ANSWER Y OR N” -enter Y or N

If Y, “NAME OF CONSTITUENT FILE:” -enter filename.
If tidal currents analysis then do east component when entire tidal current analysis performed summary table output. Otherwise program loops back to the beginning.

“OUTPUT JUST THE RESULTS OF THE ANALYSIS?”
(otherwise, it returns to the beginning.)

3 OUTPUT:

The output for this program can come in two forms. The harmonic constituent table and the statistical information following it. They can either be output to the user’s TTY, or can go directly to the line printer. The user also has the option of saving the data in the table by creating a constituent file.

If an error analysis was done, an error term will be associated with the H, G, and Kappa values for each of the 20 main frequency bands in the constituent table. The user has the option of outputting the predicted series and/or the residual series to disk. The user can also view a table of & values and their associated p and & values for the various frequencies (along with various data generated by the prediction process) by answering “N” to “OUTPUT JUST THE RESULTS
TIHARC

OF THE ANALYSIS?.

Warning: The output tidal constituent file contains ASCII phases in local epoch $\kappa$. 
APPENDIX A: A Discussion of Tidal Phase Terminology

Local epoch; $\kappa$ is the interval between high water phase of an equilibrium constituent (argument; $E = V + u$) and the following high water of the corresponding constituent of the real tide known as phase lag or epoch $\kappa$ e.g. for a particular tidal constituent

$$Y_1 = fH \cos(V = u - \kappa),$$

where

$$Y_1 = \text{height (rel to mean SL) due to constituent}$$

This is illustrated below
where

T = any instant of time under consideration (angle between meridian of
and that of observance i hours) hour angle of mean sun

M = time when equilibrium constituent argument equals zero

\( \alpha \) = the phase of the observed tide at time T is reckoned from the
preceeding high water and therefore is \((V_o + u - \kappa')\)

\( V_o + u \) = the value of the phase difference between the equilibrium tide of a
particular constituent and the beginning of the observed series.

for example, the beginning series

\( \zeta \) = the corresponding phase difference of the observed tide at the beginning of
the observed series and the time of the measured (observed) high water.

Thus

\[ \kappa = V_o + u + \zeta. \]  

(1)

Since argument formulas of all diurnal and semidiurnal constituents contain
some multiple of the hour angle (T) of the mean sun, the arguments will have
different values at different longitudes at the same instant of time. This can be
expressed in terms of the Greenwich argument value as follows:

\[ \text{local } (V + u) = \text{Greenwich } (V + r) - pL, \]  

(2)

where

\[ p = \begin{cases} 
0 & \text{long period} \\
1 & \text{diurnal} \\
2 & \text{semi-diurnal} 
\end{cases} \]

constituents

and L = west longitude of the local station.

The absolute value of the reference time depends upon time meridian used in
the locality. For example, the argument value will differ from time zone to
time zone even if the clock time of the reference time in a particular zone is
the same as the clocktime for other zones. This is accounted for by calculating
the changes in argument due to the constituent passage during the actual time
difference between zones or

\[ \text{local } (V_o + u) = \text{Greenwich } (V_o + u) - pL + \frac{as}{15}, \]  

(3)

where

a is the angular speed of constituent

s is the West longitude of the time zone meridian

\((V_o + u)\) is the value of argument at the reference time.
If we include these corrections to local \((V_o + u)\) then (3) and (1) become

\[
\kappa = \text{Greenwich} \, (V_o + u) - pL + \frac{as}{15} + \zeta \tag{4}
\]
Terminology:
The tidal height is given by
\[ h = H_0 + \sum f H \cos [at + (V_o + u) - \kappa], \]
where,
- \( h \) = height of tide at any time \( t \)
- \( H_0 \) = mean height of water relative to a datum
- \( H \) = mean amplitude of any constituent \( A \)
- \( f \) = factor reducing \( H \) to year of prediction
- \( a \) = speed of constituent \( A \) [\( t^{-1} \)]
- \( t \) = time reckoned from some initial epoch such as the beginning of year of predictions
- \( (V_o + u) \) = value of equilibrium argument of \( A \) when \( t = 0 \)
- \( \kappa \) = local epoch of \( A \)

The epoch or phase lag of a tidal constituent, \( \kappa \), is the difference between the phase of the observed constituent and the phase of its argument \((V + u)\) same time.

The relationships between commonly used nomenclature in tidal analysis are shown graphically below.

The following are some of the relationships illustrated above:
Greenwich epoch \[ G = pL + \kappa = \text{Greenwich} \ (V_o + u) + \frac{\alpha_s}{15} + \zeta \]

modified epoch \[ \kappa' = \kappa + pL \cdot \frac{\alpha_s}{15} = \text{Greenwich} \ (V_o + u) + \zeta \]

local epoch \[ \kappa = \kappa - pL + \frac{\alpha_s}{15} = \text{Greenwich} \ (V_o + u) - pL + \frac{\alpha_s}{15} + \zeta \]

where \[ p = \begin{cases} 0 & \text{longer period} \\ 1 & \text{diurnal} \\ 2 & \text{semi-diurnal} \end{cases} \]

constituents

\( s = \) longitude of local time meridian

\( L = \) west long of station for which prediction is required
APPENDIX B: "Long" Series Definition
Clearly the definition of a "long" series in connection with harmonic analysis depends upon the user-specified accuracy requirements for a set of tidal constituents. For a particular constituent, the accuracy depends on the resolution ability of the overall analysis and the amplitudes and frequency spacings of adjacent tidal lines. One of the most difficult resolution problems occurs with the $S_2$, $T_2$, $R_2$, and $K_2$, which are clustered around 2.0 cpd with 1 cpy (0.002738 cpd) separations (see figure 1).

TIHARC combines the initial Fourier analysis for each of the spectral lines with constituent interaction (due to aliasing) corrections to enhance the basic resolution of the Fourier analysis. For a "long" time series analysis TIHARC initially applies a Parzen window in the frequency domain to the input time series in order to improve the correction process. (The "short" series analysis does no windowing). The correction of line spectra and resolution improvement are possible because the aliasing of adjacent lines is well specified and iterative matrix inversion procedures can be used to make the corrections. Thus we can expect to obtain frequency resolution with time series which are shorter than those required for Fourier analysis alone. The question is how much shorter, or restated, how long must a series be so that spectral lines of interest can be resolved?

To answer this question, we have taken predicted noise-free tidal series of various lengths using our tide prediction routine, TIDHAR and performed a TIHARC "long" series analysis. The results are summarized in Table B-1 and show that a 180-day series is required for "perfect" resolution of the $S_2$, $T_2$, $R_2$, and $K_2$ tidal lines. Shorter input series are required to resolve tidal lines with more distant neighbors or having much greater amplitudes. For most practical purposes with a noise-free input series of 60d would be adequate for resolving the 25 lines analyzed by TIHARC. When an rms noise of 1 cm is added to the predicted series the results summarized in Table B-2 indicate that an input series length of d is required for a comparable accuracy. Note that the resolution of constituents with amplitudes of 0(1cm) is difficult.
## TABLE B-1. Tiharc “long” series tidal analysis of noise-free predicted series for the Cob seamount.

The tidal constituents for the input series are shown in the left hand columns and the analysis results for 30° - 360° series are shown to the right. The discrepancy between the M_1 input and analysis results is due to an inconsistency between Tiharc and TIDHAR, the tide production program.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
<td>K</td>
<td>H</td>
<td>K</td>
<td>H</td>
<td>K</td>
<td>H</td>
<td>K</td>
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<td>M₂</td>
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<td>80.95</td>
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<td>S₂</td>
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<td>115</td>
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<td>318</td>
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<td>K₁</td>
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<tr>
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<td>15.11</td>
<td>67</td>
<td>2.44</td>
<td>311</td>
<td>2.17</td>
<td>316</td>
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<td>T₂</td>
<td>1.33</td>
<td>12</td>
<td>311.17</td>
<td>28</td>
<td>11.05</td>
<td>169</td>
<td>4.24</td>
<td>38</td>
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<tr>
<td>M₅</td>
<td>2.09</td>
<td>316</td>
<td>21.62</td>
<td>188</td>
<td>2.18</td>
<td>323</td>
<td>2.08</td>
<td>316</td>
</tr>
<tr>
<td>V₂</td>
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<td>101.15</td>
<td>7</td>
<td>3.31</td>
<td>344</td>
<td>3.08</td>
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**TABLE B-2.** TIHARC “long” series tidal analysis of predictions with 1 unit rms white noise added. Otherwise the same as B-1.
Appendix C: Tidal Harmonic Constant Uncertainties

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Abstract

A method is described for estimating the uncertainties associated with the tidal harmonic constituent amplitudes and phases computed by standard harmonic analysis techniques. Adapted from the Munk and Cartwright (1966) method for estimating the uncertainties associated with transfer function estimates, it is most appropriate for use with sea level and bottom pressure observations which have relatively low levels of geophysical noise relative to tide signals.

A. Introduction

Oceanographic measurements are often dominated by fluctuations at tidal frequencies. If these tidal signals themselves are the primary interest, then the record can be analyzed by either the harmonic method developed by Lord Kelvin and George Darwin in 1867 or the response method developed by Munk and Cartwright (1966). The harmonic method has been the primary form of tidal analysis for many years, and is essentially a fourier transform computed at the frequency of the tidal line under consideration. The analysis amplitudes and phases which result from this analysis are corrected for the age of the tide and the phase of the sun and moon to obtain the amplitude and phase of the tidal constituents. This analysis result includes not only the energy in the tidal line, but also the energy in the background continuum. For the larger constituents, such as the $M_2$, this causes no problem since the signal-to-noise ratio is often more than 40 db. However, for the smaller constituents (such as $R_2$, $O_1$, etc.) the signal-to-noise ratio approaches 1, and the analysis uncertainty is greater. The analysis can be improved by using a longer record, which permits the use of a narrower bandwidth, so that less background noise is included with the analysis of a particular line. These considerations should be included in estimating the error associated with that analysis.

Two versions of the harmonic method are used by our research group at the University of New Hampshire (UNH) to analyze for tides. The first is a modified
version of the National Ocean Survey (NOS) algorithm (Dennis and Long, 1971) which directly computes the constituent amplitudes and phases for the \( K_1, O_1, M_2, N_2, S_2, M_4, S_4, M_6, S_6, \) and \( M_8 \) lines and infers another 15 less important constituents using equilibrium tidal theory. The effective bandwidth associated with this analysis is \( \Delta f_e = \Delta T_A = 1/T_s \), where \( T_s \) is the length of the series under analysis. The second UNH version of the harmonic method does the Fourier analysis at the frequencies of all 25 harmonic constituents analyzed for by the NOS routine, and works best on sea level or bottom pressure series longer than 2 months. The effective bandwidth of this analysis is \( \Delta f_e = 3.71 \Delta T_A \), which is appropriate for the Parzen windowing used.

Munk and Cartwright (1966) present a method for estimating the uncertainty associated with tidal admittances computed in their response analysis. We have adapted the formalism of Munk and Cartwright (1966) for use with the harmonic analysis of tides and present the details in this paper. In Section B we show that the noise associated with computing the constituents by the harmonic method is the same as that associated with computing the transfer function estimates. In Section C, the Munk-Cartwright method for estimating uncertainties in transfer function calculations is adapted to the harmonic analysis of the tidal constituent amplitudes and phases.

\section{B. Theoretical Considerations}

Assume that the observed tide, \( y(t) \), is the outcome of a linear process characterized by the transfer function, \( T(f) \), and corresponding weighting function, \( \Gamma(t) \), with random noise, \( n(t) \), at the output as shown below.

\begin{equation}
\begin{array}{c}
x(t) \overrightarrow{\Gamma(t)/T(f)} \overrightarrow{v(t)} \overrightarrow{\otimes} \overrightarrow{y(t)} \\
\uparrow n(t)
\end{array}
\end{equation}

The input \( x(t) \) in this case is the noise-free astronomical forcing, which is specified in terms of the equilibrium tide. The measured response of this system, \( y(t) \), is the sum of the noise-free tidal response, \( v(t) \), which is coherent with \( x(t) \), plus the random noise at the output, \( n(t) \), according to

\begin{equation}
y(t) = v(t) + n(t) .
\end{equation}

The tidal response, \( v(t) \), is described here in terms of the harmonic constituents, \( H_n \) and \( K_n \), at each frequency, \( f_n \), of the equilibrium tide. We seek an estimate of the uncertainty in estimates of \( H_n, K_n \) in the presence of noise. In the
frequency domain (1) becomes

\[ G_y(f) = G_v(f) + G_n(f) \]

where the G's are one-sided spectral density functions as defined by Bendat and Piersol (1971). The output can be expressed as the convolution

\[ \nu(t) = \int_{0}^{\infty} \Gamma(\tau) \cdot x(t-\tau) d\tau \]

where \( \tau \) is the time lag. For the noise-free part of our system, the convolution theorem permits us to write

\[ G_{xy}(f) = T(f) \cdot G_x(f) \quad (2) \]

Here, \( G_{xy}(f) \) is the complex, one-sided, cross-spectral density function,

\[ G_{xy}(f) = C_{xy}(f) + Q_{xy}(f) \]

where \( C_{xy}(f) \) and \( Q_{xy}(f) \) are the co and quadrature spectrum, respectively. In polar notation (2) becomes

\[ |G_{xy}(f)| e^{-i\alpha_{xy}(f)} = |T(f)| e^{-i\Phi(f)} G_x(f) \]

where

\[ |G_{xy}(f)| = \left( C_{xy}^2(f) + Q_{xy}^2(f) \right)^{\frac{1}{2}} \]

and

\[ \alpha_{xy}(f) = \tan^{-1}\left\{ Q_{xy}(f)/C_{xy}(f) \right\} \]

Thus

\[ |G_{xy}(f)| = |T(f)| G_x(f) \]

and

\[ \alpha_{xy}(f) = \Phi(f) \]

Since \( G_x(f) \) is noise-free, the uncertainties in computing estimates of \( G_{xy}(f) \) and \( \alpha_{xy}(f) \)s or \( |T(f)| \) and \( \Phi(f) \)s are equivalent. Unfortunately, we cannot compute \( G_{xy}(f) \) and \( \alpha_{xy}(f) \) in our noisy system and are restricted to the relation corresponding to (2), which is \( G_{xy} = \bar{T}(f) \cdot G_x(f) \), where \( \bar{T}(f) \) is an estimate of the transfer function based on the measured output.

There are several references on estimating uncertainties for \( |T(f)| \) and \( \Phi(f) \). From the Appendix B of Munk and Cartwright (1966), transfer function estimates, \( \bar{T}(f) \), are written in terms of the true transfer function, \( T(f) \), and a random noise according to

\[ \bar{T}(f) = G_{xy}(f)/G_x(f) = T(f) + \varepsilon = |T(f)| e^{-i\Phi(f)} + \varepsilon \quad (3) \]
Here, \( \varepsilon = G_n(f)/G_x(f)e^{-i\tau(f)} \) is a complex random variable, whose real and imaginary parts have approximate normal probability distributions with zero mean and the variance in the fundamental frequency band, \( \Delta f \), is given by

\[

\sigma^2 = \frac{|G_n(f)\Delta f|/2p}{|G_x(f)\Delta f|}.

\]

(4)

Here, \( p = \Delta f x T_x \) is the number of degrees of freedom of the estimate, for which \( \Delta f \) is the effective bandwidth used in the spectral computation, and \( T_x \) is the length of series. (Note that \( G_n(f)\Delta f/2p \) is the mean square error in estimating a mean square value in the presence of band-limited white noise.) We normalize \( \sigma^2 \) by the magnitude of the true transfer function \( |T(f)| = |G_{xx}(f)|/G_x(f) \) and find

\[

\sigma^2 = \frac{\sigma^2}{|T(f)|} = \frac{G_n(f)\Delta f}{2p|G_{xx}(f)|\Delta f}

\]

(5)

as Munk and Cartwright have. They go on to compute the probability density function and the 95% confidence limits for the normalized variables \( \rho = \tilde{T}(f)/T(f) \) and \( \theta = \tilde{\Phi} - \Phi \) in terms of \( \sigma \) (see Figure 1).

The Munk and Cartwright results in Figure 1 are useful to us if we can estimate \( \sigma^2 \) in terms of spectral estimates other than the cross-spectrum \( |G_{xx}(f)| \), which, of course, we do not compute. In order to resolve the problem, consider the harmonic analysis of a sine wave of amplitude \( H \) in the presence of low-level white noise. The energy density amplitudes of the sine wave at frequency \( f_n \) and noise in the bracketing tidal analysis fundamental frequency bandwidth \( \Delta f_{TA} \) are shown in the sketch below.

\[

\frac{H^2(f_n)}{2\Delta f_{TA}}

\]

\[

G_n

\]

\[

\Delta f_{TA}

\]

\[

f_n

\]

If the variance ratio \( G_n\Delta f_{TA}/H^2 << 1 \) (i.e., large tidal-signal-to-noise ratio), then the total variance within \( \Delta f_{TA} \) can be approximated as

\[

|G_{xy}(f_n)|\Delta f_{TA} = \frac{H^2(f_n)}{2} + G_n\Delta f_{TA}

\]

(6)

\[

\sim \frac{H^2(f_n)}{2} = |G_{xx}(f_n)|\Delta f_{TA}

\]

(7)
Figure 1. The 95% confidence limits for the normalized amplitude, $\rho = \text{sample admittance/true admittance}$, (left) and error phase, $\theta = \text{sample phase - true phase}$, for stated values of the noise parameter $\sigma$ (from Munk and Cartwright, 1966).
which is bandwidth-independent, while the noise variance, $G_n \Delta f_{TA}$, is bandwidth-dependent. If we define an average background noise energy, $\bar{G}_n$, in the tidal analysis bandwidth, $\Delta f_{TA}$, then a noise parameter estimate, $\hat{\sigma}^2$, can be derived from (5) by substituting appropriately for $p$ and using (7) so that

$$\hat{\sigma}^2 \sim \frac{\bar{G}_n \Delta f_{TA}}{T_s \Delta f_c \Delta f_{TA}^2 (f_n)}.$$  \hspace{1cm} (8)

C. Application

As we have seen, the uncertainty in the estimates of tidal constituent amplitude and phase is related to the signal-to-noise ratio in the spectral frequency band associated with a particular constituent. The most important tidal constituent frequencies are clustered within ±4.5 cycles per lunar month (cpm) (or 0.013726 cph) of the lunar daily, $f_{M1}$, and lunar semidiaily, $f_{M2}$, frequencies. Our approach is to estimate the average value of noise energy density, $\bar{G}_n$, in the 9 cpm frequency bands, $\Delta f_i$ (i = 1 or 2), bracketing $f_{M1}$ and $f_{M2}$. [$\Delta f_1 = 0.04025 \pm 0.00687$ cph (0.04712 to 0.03338 cph) and $\Delta f_2 = 0.08051 \pm 0.00687$ cph (0.08738 to 0.07364 cph).]

The band average noise energy density $\bar{G}_n$ is found as follows. First, the predicted tidal series is subtracted from the observed series to produce a residual series. Then, a standard FFT routine is used to compute raw spectral energy estimates, $G_n(f_j)$, for frequencies $f_j = f_0 + j \Delta f$ (where $j = 1, 2, \ldots k/2$, $\Delta f = 1/T_s$, and $T_s = k \Delta t$). Then the total noise variance in $\Delta f_i$ is found according to

$$N_i = \sum_{\Delta f_i} n_j = \sum_{\Delta f_i} G_n(f_j) \Delta f,$$

where only the $n_j$ within $\Delta f_i$ are summed. The corresponding average value of noise spectral energy density is

$$\bar{G}_{ni} = \frac{N_i}{\Delta f_i}.$$

The confidence in this estimate can be assessed. We assume a white noise in the vicinity of the tidal energy bands and therefore each of the spectral estimates of noise $G_n$ are statistically independent and associated with 2 degrees of freedom. Thus the total degrees of freedom (and hence the confidence) associated with $\bar{G}_{ni}$ is

$$\text{degrees of freedom} = 2 \left( \frac{\Delta f_i}{\Delta f} \right) = 2T_s \Delta f_i = \frac{18T_s}{655.7h}.$$

Therefore, for tidal series for which $T_s > 1$ month $\sim$ 656 hours, we have about 18 degrees of freedom and reasonable confidence in our estimates of $\bar{G}_{ni}$. 
An example of this procedure can be seen in the spectra of the total and residual signals from a bottom pressure station in 77 m of water on the south flank of Georges Bank shown in Figures 2 and 3. The average diurnal and semidiurnal noise energy densities are indicated in Figure 3. For the $M_2$ tidal line, the average noise spectral energy density $G_n = 3.1 \times 10^{-2}$ dbar$^2$/cph. For the inferred harmonic analysis of this record [$\Delta f_{TA} = (1250 \text{hr})^{-1}$], the amplitude of the $M_2$ constituent was $H = 0.397$ dbar. From (8) an estimate of the noise parameter is

$$\sigma^2 \approx \frac{3.10 \times 10^{-2}}{(0.397)^2 \text{ dbar}^2 \cdot 1250 \text{ cph}^{-1}} = 1.56 \times 10^{-4} .$$

From Figure 1, based on $\hat{\sigma}$, the uncertainty in the amplitude is $\pm 2\%$ (or $\pm 0.08$ dbar), and the uncertainty in the phase is $\pm 1.0$ degree.

Acknowledgements

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References


Figure 2. An FFT (with no averaging) of a 1250-hour section of the bottom pressure on Georges Bank showing the dominance of the semidiurnal and diurnal tidal signal at this site. The presence of a small $M_4$ and $M_6$ contribution is also indicated.
Figure 3. An FFT (with no averaging) of a 1250-hour section of the bottom pressure residual (astronomical tides removed) showing significantly reduced energy at the semidiurnal and diurnal frequencies. The energy, which does remain at the tidal frequencies, is not coherent with the astronomy and probably is due to the baroclinic tidal response. The average energy density, as computed according to methods described in the text, is indicated in the two tidal bands.