Exercize 7. Massachusetts Internal Tides

Given: Spectra of Idealized Internal Waves

Density stratification of the ocean water column enables the vertical displacement of isopycnals (i.e. surfaces of constant density). In theory, the maximum allowable frequency for these isopycnal displacement oscillations is the local buoyancy frequency \( N \) (where \( N^2 \propto \) stratification strength); the minimum is the local inertial frequency \( f_i \) (which varies with latitude).

The idealized autospectrum of a typical mid-ocean isopycnal displacement time series \( z(t) \) due to internal gravity waves in the has been approximated by Garrett and Munk (1972) as

\[
G_{zz}(f) = A/f^2 \quad \text{for} \quad f_i = 0.04\text{cph} < f < N = 2\text{cph} \\
= 0 \quad \quad \text{for} \quad f > 2\text{cph} \quad \text{and} \quad f < 0.04\text{cph}
\]

where the value of \( A \) depends on the ocean depth at which the observations are made; typically \( A = 1 \ (\text{m}^2\cdot\text{cph}) \) at a depth of 1000m.

(a) What are the units of \( G_{zz}(f) \) in the equation above?

\[
[G_{zz}(f)] = [A] / [f^2] = \text{m}^2\cdot\text{cph} / \text{cph}^2 = \text{m}^2 / \text{cph}
\]
(b) What is the variance of the idealized mid-ocean internal waves (IW) within a frequency band $\Delta f = 0.007142857$ cph that brackets the $M_2$ semidiurnal frequency of 0.0805 cph?

**Upper frequency limit** $f_u = 0.0805 + \Delta f/2 = 0.0805 + 0.003571428 = 0.0841$ cph;

**Center frequency** $f_c = 0.0805$ cph;

**Lower frequency limit** $f_l = 0.0805 - \Delta f/2 = 0.0805 - 0.003571428 = 0.0769$ cph;

Variance $M_2 = \int_{f_l}^{f_u} G_{zz}(f) df = -A f^{-1} |_{f_l}^{f_u} = 1.113 \text{ m}^2$

What are the corresponding standard deviation?

$$\text{Std. Dev.} = \sqrt{\text{Variance } M_2} = 1.055 \text{ m}$$

and the equivalent amplitude of an internal wave represented by that variance.

For a sine wave with zero mean value, amplitude $a$, and phase $2\pi f_c t = \alpha$, the

Variance $M_2 = \text{mean square} = (2\pi)^{-1} \int_{-\pi}^{\pi} a^2 \sin^2 \alpha d\alpha = (2\pi)^{-1} a^2 \int_{-\pi}^{\pi} (1 - \cos 2\alpha)/2 d\alpha = a^2/2$

Therefore $a_{equ} = \sqrt{2 \times \text{Variance } M_2} = \text{Std. Dev. } M_2 \times \sqrt{2} = 1.492 \text{ m}$

(c) What is the total IW variance (and corresponding standard deviation) in the range $0.056 \text{ cph} < f < 0.500 \text{ cph}$?

Wideband $G_{zz}(f)$ Variance $= \int_{0.056 \text{ cph}}^{0.500 \text{ cph}} G_{zz}(f) df = -A f^{-1} |_{0.056}^{0.500} = 15.857 \text{ m}^2$

$$\text{Std. Dev.} = \sqrt{\text{Wideband } G_{zz}(f) \text{ Variance}} = 3.982 \text{ m}$$

What is the percentage of the total variance that is due to the $M_2$ semidiurnal frequency band variability?

$$\% = \frac{1.113 \text{ m}^2}{15.857 \text{ m}^2} = 7.02$$
Problem 1  Statistical Analysis of Observed Internal Waves

Hourly-averaged time series of pairs of temperature/conductivity measurements were made at 4, 10, 25, 45 and 60m depths, respectively, in 87 m of water from a mooring deployed in Stellwagen Basin (Massachusetts Bay; $f_i = 0.056$ cph; $1 \text{ cph} < N < 30 \text{ cph}$) at $42^\circ 21.2' N$, $70^\circ 24.3' W$ from July 8 1990 through 10 August 1990. These measurements were used to compute a time series of density anomaly (i.e., sigma-theta) from which a set of isopycnal displacement series were estimated via linear interpolation.

Here we consider 3 time series of 856 hourly samples of isopycnal displacement for sigma theta = 23.70; 25.30; and 25.55 (files called i237.sum, i253.sum, and i2555.sum respectively) are in /hosts/XXXXXXXXXXXXX/exam_I.dir. It is clear in Figure 1 that internal waves at tidal frequencies – internal tides - dominate the 2 deeper records tides. But there is other variability also.

![Stellwagen Basin Isopycnals (*.sum) Jul–Aug 1990](image)

**Figure 1.** The *.sum isopycnal depth records.
The bulk statistics of the 3 *.sum series on a 0 to -100m depth versus time are:

<table>
<thead>
<tr>
<th>name</th>
<th>maximum (m)</th>
<th>minimum(m)</th>
<th>mean (m)</th>
<th>std dev(m)</th>
<th>var(m^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>i237.sum</td>
<td>-.595848E+01</td>
<td>-.186592E+02</td>
<td>-.135789E+02</td>
<td>1.72931</td>
<td>2.9905</td>
</tr>
<tr>
<td>i253.sum</td>
<td>-.234051E+02</td>
<td>-.453628E+02</td>
<td>-.315865E+02</td>
<td>5.62363</td>
<td>31.6252</td>
</tr>
<tr>
<td>i2555.sum</td>
<td>-.308666E+02</td>
<td>-.687543E+02</td>
<td>-.493001E+02</td>
<td>7.98109</td>
<td>63.6978</td>
</tr>
</tbody>
</table>

(a) In an effort to focus on the internal tidal wave frequency band, you are asked to apply a high pass filter to these series. Use\texttt{\_filter} [36-term, lanczos high pass filter with a cutoff frequency = 0.056 Nyquists] on the three isopycnal series to produce the corresponding *.hs series.

(1) What is the dimensional cutoff frequency of the high pass filter?

(2) Why is it necessary to remove the low frequency variability?

(b) Compute the bulk statistics (\texttt{\_lstats}) and plot the *.hs series using the same format as in (a).

How much and what percentage of the original variance was due to the low frequency variability?

(c) Compute (\texttt{\_specter} with a fundamental frequency band of 0.007142857cph) to compute the autospectrum of the filtered 25.30 isopycnal displacement series.

(1) Plot the autospectrum on log-log axes of dimensional spectral energy density and frequency...complete with the appropriate 95% uncertainty limits for the relevant degrees of freedom (DOF).

(2) Assuming that \( f = 0.007142857 \)cph is in the center of the first harmonic, at what harmonics (and frequencies) do you find statistically significant peaks in the spectrum?

(3) At what frequency band do you find most of the variance?

(d) Use cross-correlation function analysis to briefly look at the structure of these isopycnal displacement time series.

(1) Specifically, compute the trio of cross-correlations (50 lags) of both the 3 *.hs series using \texttt{\_correl} (option 3 so you can obtain the “long lag”
Problem 2  Comparison of Theoretical and Observed Internal Wave Variability

The analysis results of Problem 2 show that the observed internal waves in Massachusetts Bay have a strong tidal frequency component that the mid-ocean internal waves do not. In order to make a meaningful comparison between the 2 regimes, we must “remove” the internal tides from the Stellwagen basin records.

(a) Remove the tides by applying an optimal filter – tidal harmonic analysis (l_tiharc98) in this case- to the i253 hs series.

(b) Compute bulk statistics and plot the detided, filtered series.

What, if any, visual differences are there?
If there are visual differences, explain using the low pass filter response if appropriate.

(c) Compute the autospectra (l_specter) for the series and plot on the same format as above.

(d) Compare the Stellwagen Basin and mid-ocean internal wave variance in the frequency band containing $f = 0.0805 \text{cph}$ in terms of the (a) variance, (b) standard deviation, (c) equivalent amplitude (d) and percentage of the $0.056 \text{cph} < f < 0.500 \text{cph}$ frequency band energy.

Problem 3.  SISO Transfer Function: Internal Tidal Forcing

The interaction of tidal currents and bathymetry are known to generate internal tidal waves. Above you have analyzed the internal tidal signals in Stellwagen Basin above. Here we ask you to compute the Single Input/Single Output (SISO) transfer function and the associated noise between a relevant current measurement and the output “signal”. For these purposes an hourly-averaged time series of current in 20 m of water from a mooring deployed on Stellwagen Bank is provided in /hosts/XXXXXXXXXXXXXX/exercize7.dir. Use the fundamental frequency bandwidth from Problem 1 above and estimate uncertainty limits.