APPENDIX A: Physics Review

"The heart of physics is an understanding of the quantitative relationships between the position and motion of mass and the forces causing changes in that motion."

The following is a brief refresher describing the tools we will use in developing these quantitative relationships for ocean currents and wastes. This presentation is not meant to be complete so if some of the ideas are unfamiliar you are referred to standard physics texts for clarification and elaboration.

Units

The fundamental quantities of distance, time and mass (usually the meter, second and kilogram respectively) are defined in terms of standards which are maintained and preserved in the International Bureau of Standards. Several systems including the mks and cgs have been devised for finding multiples and submultiples of the standard units.

The units for other derived quantities such as force, torque, density, etc. are determined unambiguously from these fundamental quantities.

Tools

Scalars: Quantities which have a magnitude only;

For example, temperature and pressure.

Vectors: Quantities that have both magnitude and direction;

For example, the vector displacement of a particle is described by its scalar distance and direction.

Equivalent vectors have equal magnitudes and the same directions;

Vectors $\vec{a}$ and $\vec{b}$ in Figure A1 are equivalent vectors with equal magnitudes $|\vec{a}|$ and $|\vec{b}|$ and the same directions.

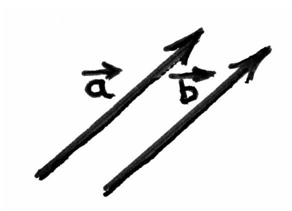

Figure A1. Equivalent vectors

Vector Operations include vector addition, subtraction and multiplication.

Two examples of vector addition (see Figure A2).

$$\vec{a} + \vec{b} = \vec{c}$$
$$\vec{b} + \vec{a} = \vec{c}$$

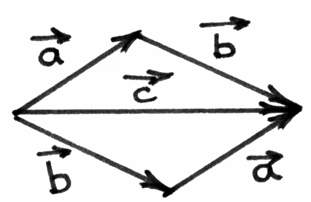

Figure A2. Vector addition

An example of vector subtraction (see Figure A3).

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \vec{c}$$

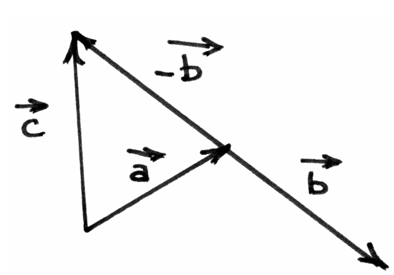

Figure A3. Vector subtraction

Defining a coordinate system, like the 2-dimentional x-y system in Figure A4, enables vector $\vec{a}$ to be resolved into its scalar components along the x and y axes according to:

$$a_x = |\vec{a}| \cos q$$
$$a_y = |\vec{a}| \sin q$$

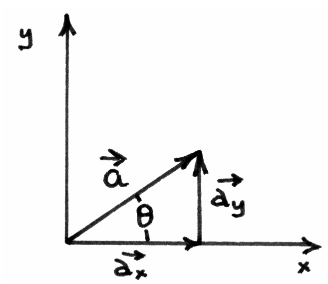

Figure A4. The resolution of vector $\vec{a}$ into its scalar components.

To reconstruct vector $\vec{a}$, first define unit vectors $\vec{i}$ and $\vec{j}$ (in the x and y directions respectively; each with magnitudes of 1), second multiply the unit vectors by the appropriate component magnitudes a_x and a_y respectively, and third do the vector addition $\vec{a} = a_x \vec{i} + a_y \vec{j}$ as shown in Figure A4:

Vector Multiplication

There are two kinds of vector multiplication; namely **scalar multiplication** and the **vector multiplication** of two vectors.

The scalar product of two vectors is

$$\vec{a} \bullet \vec{b} = \vec{c} = |\vec{a}| \cos q \, |\vec{b}| = |\vec{a}||\vec{b}| \cos q \qquad .$$

The graphical presentation of the above operation in Figure A5 shows that a scalar product $\vec{a} \bullet \vec{b}$ is either the (a) component of $\vec{a}$ along $\vec{b}$ times $|\vec{b}|$ or the (b) component of $\vec{b}$ along $\vec{a}$ times $|\vec{a}|$.

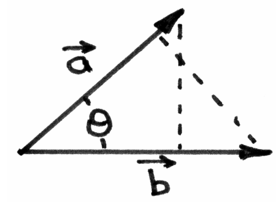

Figure A5. The components of a scalar product of 2 vectors.

Note that if $\vec{b} = \vec{i}$, then the scalar product operation

$$a \bullet \vec{i} = a_x$$

produces the vector component in that direction; i.e., vector resolution.
A well-known example of a scalar product is the particle kinetic energy

$$K.E. = 1/2\,m(\vec{v} \bullet \vec{v}) = 1/2\,m(v_x^2 + v_y^2 + v_z^2).$$

The vector product of two vectors is defined as

$$\vec{a} \text{ x } \vec{b} = \vec{c} = |\vec{a}| \sin q\, |\vec{b}| \frac{\vec{c}}{|\vec{c}|} \quad ‘$$

where $\frac{\vec{c}}{|\vec{c}|}$ is the unit vector that is perpendicular to plane of $\vec{a}$ and $\vec{b}$. The so-called right hand rule (for an advancing screw) determines the sense of $\vec{c}$ as illustrated in

$$\vec{a} \text{ x } \vec{b} = \vec{c}$$

or

$$\vec{b} \text{ x } \vec{a} = -\vec{c}$$

A well-known example of a vector product is the torque, $\vec{T}$ about at point

$$\vec{T} = \vec{r} \times \vec{F} \qquad ,$$

where $\vec{F}$ is the force acting at $\vec{r}$ is the radius vector from the point.

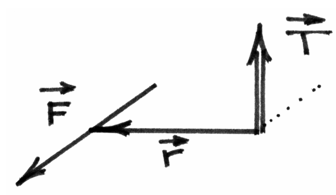

Figure A6. The depiction of vector torque

Other Tools

Consider f(x) – an <u>explicit function</u> of x. The <u>differential</u> of f(x) is $\frac{df}{dx}$. If x = x(t), then f(x) is an <u>implicit function</u> of t and the differential $\frac{df}{dt}$ can be computed using the <u>chain rule</u> according to

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

For example, if $f(x) = x^2$ and $x = t^2$, then $\frac{df}{dt} = 2x \bullet 2t = 4t^3$.

Consider f(x, y, z) - an explicit function of variables x, y and z. The total differential of f(x, y, z) is

$$df = \left.\frac{\partial f}{\partial x}\right|_{y,z} dx + \left.\frac{\partial f}{\partial y}\right|_{x,z} dy + \left.\frac{\partial f}{\partial z}\right|_{x,y} dz \quad ,$$

where $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are <u>partial differentials</u> with respect to one of the independent variable computed while the other variables are held constant.

Combining the calculus and vector considerations from above, we define a gradient vector. Specifically, if $f = f(x, y, z)$ is a scalar, then its gradient ois

$$\nabla f = \nabla f = \frac{\partial f}{\partial \mathrm{x}}\vec{\mathrm{i}} + \frac{\partial f}{\partial \mathrm{y}}\vec{\mathrm{j}} + \frac{\partial f}{\partial \mathrm{z}}\vec{\mathrm{k}} \qquad ,$$

where the del operator $\nabla \equiv \frac{\partial}{\partial \mathrm{x}}\vec{\mathrm{i}} + \frac{\partial}{\partial \mathrm{y}}\vec{\mathrm{j}} + \frac{\partial}{\partial \mathrm{z}}\vec{\mathrm{k}}$.

Mechanics of a Particle

The field of mechanics is divided into kinematics and dynamics.

Kinematics deals with the description of the relation of position, velocity and acceleration.

Dynamics deals with the relation of the motion with the forces causing it.

Particle Kinematics

Consider the kinematics of a particle starting at position **a** at time $t = t_1$ and following the curved path shown in Figure A7. The displacement for the particle between times t_1 and t_2 is the vector $\vec{\mathrm{x}}$.

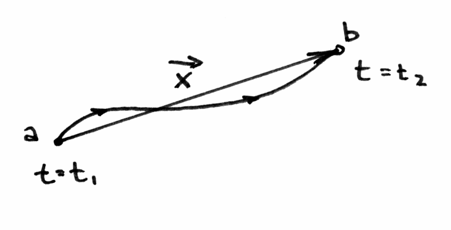

Figure A7. Particle trajectory and displacement

The average particle velocity is

$$\vec{V}_a = \frac{\vec{x}}{\Delta t} \qquad ,$$

where $\Delta t = t_2 - t_1$.

The instantaneous particle velocity is

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t} = \frac{dx}{dt} \qquad ,$$

where $\Delta \vec{x}$ is a "small displacement" which occurs over a "small time" interval, Δt. The magnitude of the velocity is the speed

$$s = \left| \frac{d\vec{x}}{dt} \right|$$

which is a scalar quantity.

Similarly, the instantaneous particle acceleration is

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{x}}{dt^2}$$

Examples: Consider two cases of constant magnitude acceleration.

1) Rectilinear Motion

Consider one-dimensional motion with a constant initial acceleration (see Figure A8); Thus

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_o \qquad ,$$

where $\vec{a}_o$ is $\vec{a}$ at t = 0 and due to a change in magnitude of the velocity.

Now find $\overrightarrow{v(t)}$ from

$$\vec{v} = \int \vec{a}\, dt = \vec{a}_o t + \vec{v}_o$$

where $\vec{v} = \vec{v_o}$ at $t = 0$.

Now find $\overrightarrow{x(t)}$ from

$$\vec{x} = \int \bar{v}\, dt = \frac{\vec{a}_o t^2}{2} + \vec{v}_o t + \vec{x}_o$$

where $\vec{x} = x_o$ at $t = 0$.

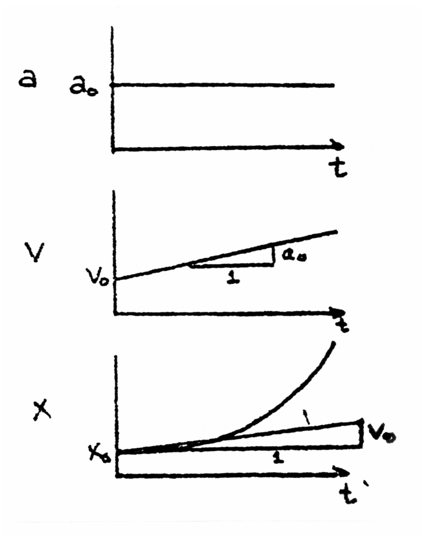

Figure A8. Particle kinematics relating instantaneous displacement velocity and acceleration.

2) Circular Motion (uniform)

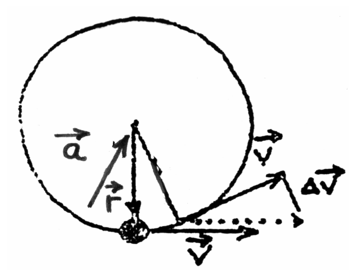

Figure A9. Uniform circular motion of a particle.

The magnitude for the acceleration is

$$|\vec{a}| = \frac{\vec{v} \bullet \vec{v}}{|\vec{r}|} = \frac{v^2}{r} \qquad ,$$

while the direction of the acceleration is $\frac{-\vec{r}}{|\vec{r}|}$.

Here the acceleration is due and direction change of a constant magnitude velocity.

Elements of Dynamics of a Particle

In this course the basis for the dynamics we consider will be Newton's First and Second Laws of Motion.

Newton's First Law describes undisturbed rest or motion and states in effect that:

"If left undisturbed, a body will tend to stay at rest if originally at rest or if originally moving it will continue at a constant velocity."

Newton's Second Law describes the way a body changes its motion if it is disturbed and states that:

"The time rate of change of a quantity called <u>momentum</u> is proportional to the applied resultant force".

Symbolically Newton's Second Law is given by

$$\vec{F} = \frac{d(m\vec{v})}{dt} \quad ,$$

where $\vec{F}$ is the resultant of forces applied to a mass m with momentum= $m\vec{v}$; in which $\vec{v}$ is its velocity. (Note that here mass is the <u>quantitative measure of inertia</u>. For example, while the weight of a particular object would be greater on Earth than on Mars, the same force would be required to accelerate that mass on both planets).

Usually m = constant and since acceleration $\vec{a} = \frac{d\vec{v}}{dt}$, the familiar form of Newton's Second Law can be written

$$\vec{F} = m\vec{a} \quad .$$

This vector form of Newton's Second Law can be reduced to its three scalar component equations

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_z = ma_z$$

each of which must be satisfied independently for all time.

Now let's explore the solution of following classical physical problem.

<u>Given</u>: A frictionless block is at rest at x = 0 at t = 0 on an inclined plane (Figure A10). (Note: the choice of coordinate system simplifies the problem). Describe its subsequent

position as a function of time. To solve the problem:

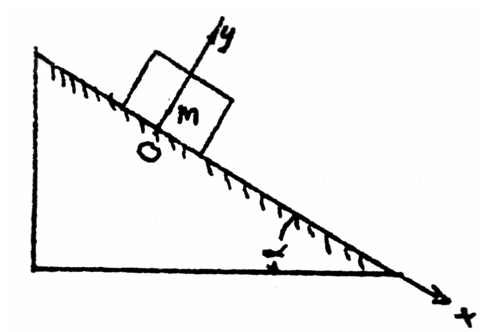

Figure A10. A frictionless block sliding on a sloping surface of angle α .

First, construct a free-body diagram for the problem (see Figure A11), in which $m\vec{g}$ is the body weight and $\vec{N}$ is the normal force applied by the inclined plane to the block.

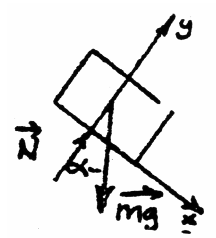

Figure A11. A "free body" diagram of the frictionless block sliding down an inclined plane.

Second, apply Newton's 2nd Law in the 2 coordinate directions.

In the x-direction

$$F_x = mg \sin\alpha = ma_x$$

or

$$a_x = g \sin\alpha \quad ,$$

which gives the constant acceleration of the block down the inclined plane;

In the y-direction, the block does not move (or accelerate), thus

$$F_y = ma_y = N - mg \cos\alpha = 0$$

or

$$N = mg \cos\alpha$$

Once the results of Figure A8 are applied with $a_o = a_x$, the problem is solved.

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DOING PHYSICS PROBLEMS

Dimensions and Dimensional Arguments

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When doing any physics problem <u>always</u> keep in mind the dimensional nature of physical data and check accordingly.

<u>Several Rules</u>

1. Do not add quantities with different units;
   (e.g. cm and m/sec or even (carelessly) cm and m).
2. You can not take the <u>sine of 10 cm</u>.
   All mathematical functions must have dimensionless arguments.
3. The dimensions of a quantity that is a product of two dimensional quantities are the product of the dimensions.
4. If your "formula " does not give answers with the proper dimensions, then <u>it is not right</u>. (For example, given an object traveling at speed C cm/sec for T days, the distance it travels can not be C/T, which has dimensions length/time^2, but must be C·T with the additional days to seconds conversion factor.)

Guidelines

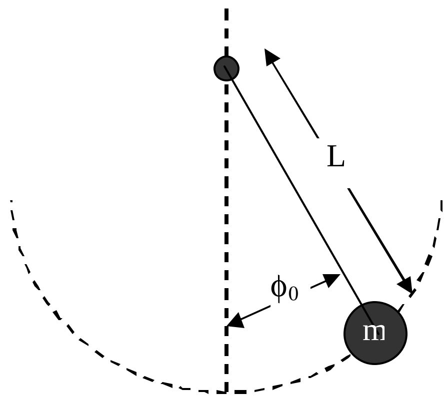
1. Check formulae first using "abstract" dimensions length, time, mass, etc.;

L	=	C	T	
Length	=	length/time	time	= length

Correct!

2. Check to see if a units conversion factor (e.g. 60 sec/min) is needed.

Frequently these ideas can give a first approximation to an answer with very little work. For example, suppose we wish to know the period T of a pendulum with a length L = 10 cm and a mass m = 100 gm in a gravitational field in which g = 980 cm/sec^2.



The available physical quantities and their units are:

[L] = length; [g] =length/time^2 [m] = mass

[θ_o - initial angle] = radians; [T] = time

The only way we can construct the period T with units of time is

$$T = \sqrt{(L/g)}\, f(\theta_o) \qquad ,$$

where f is some unknown mathematical function [here's where physics is necessary]. Essentially for free we have learned that T does not depend on m and must be proportional to $\sqrt{L}$. If f is about 1, then T is about $\sqrt{10/980}$ sec or about 0.1sec. In fact, f ~ 2π.

APPENDIX A PROBLEMS

1. Long-term current meter measurements located on the continental shelf south of Nova Scotia show that average near-surface currents are <u>westward</u> at 20 cm/s.

    a) Calculate the alongshore (parallel to shore) and cross-shore (perpendicular to shore) components of this current taking into account the fact that the Nova Scotia coastline is oriented along a <u>65 degrees True</u> compass heading. Make sure to define and sketch your coordinate systems and velocities. Show all work.

    b) A Canadian Coast Guard search and rescue team is searching for a fishing vessel which went down 100 km off the coast of Nova Scotia in the same area described above. Based on the average current velocity in the region how far <u>along the coast</u> and <u>perpendicular to the coast</u> should the search team look for survivors 24 hours after the vessel has sunk? Show all work.

2. A velocity field may be defined as follows:

$$u(x,y,z,t) = 5t^2 + 3x + 2y$$
$$v(x,y,z,t) = 0$$
$$w(x,y,z,t) = 0$$

    a) Compute an expression for the total derivative of the above velocity field. Show all work.

    b) Compute the total derivative of the velocity field at t=2, x=3, and y= 3. Show all work.

    c) What is the ratio of the "local" acceleration to the "advective" accelerations?

3. A typical change in the sea surface height across the Gulf Stream is approximately 1 m. Given that the Gulf Stream is approximately 100 km in width, what is a typical sea surface slope (in degrees please!) across the Gulf Stream. Draw a diagram as part of your answer and show all work.

APPENDIX B: Derivation of the Coriolis Force

Consider a vector $\vec{A}$, which changes with time at a rate $\frac{d\vec{A}}{dt}|_{rot}$ as viewed by an observer in the Earth's rotating (accelerating) frame of reference. However, even if $\frac{d\vec{A}}{dt}|_{rot} \equiv 0$, then the vector $\vec{A}$ will be seen to change direction by an observer in an inertial frame of reference in outer space because it is fixed to a rotating Earth. Figure B1 shows a vector $\vec{A}$ (which is tilted at an angle y relative to the Earth's rotation vector $\vec{\Omega}$) at time t and time t + dt. As the Earth rotates through the angle a

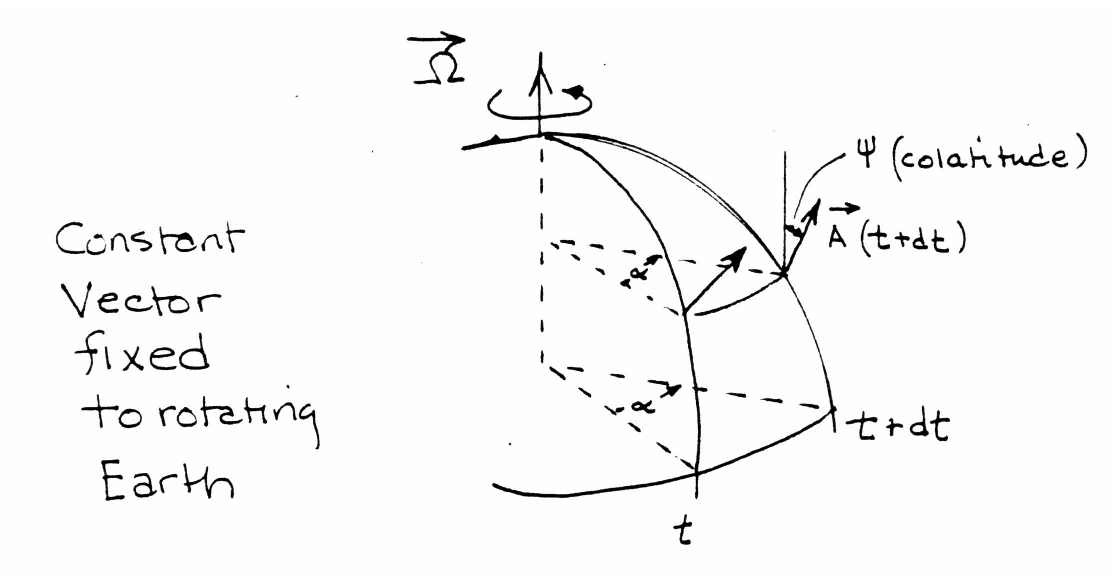


Figure B1 Vector displacement on a rotating earth. The meridional plane of $\vec{A}$ is translated through an angle a in time dt

the component of $\vec{A}$ parallel (11) to the axis of rotation $\vec{\Omega}$ is unchanged in magnitude and direction. However, the component of $\vec{A}$ perpendicular ($\perp$) to $\vec{\Omega}$, $\vec{A}'$ in Figure B2, changes direction as shown.

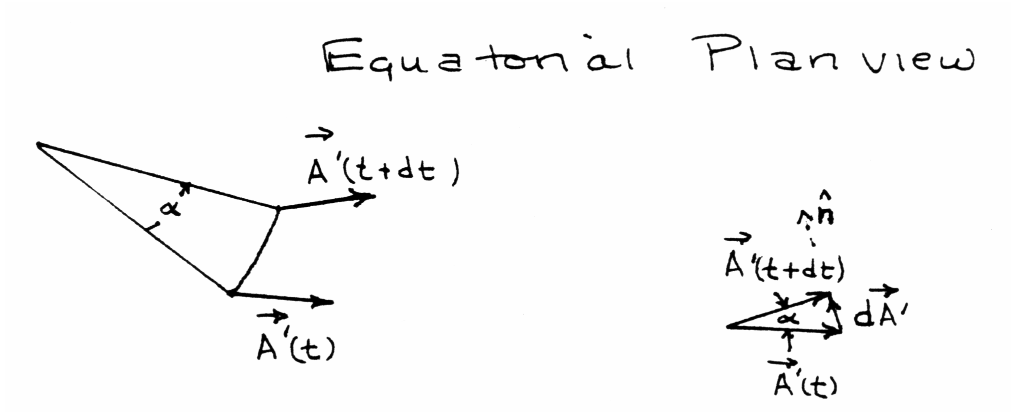


Figure B2. The change in direction of the vector $\vec{A}'$, due to earth rotation through angle a in time dt.

The differential change in $\vec{A}'$ is

$$d\vec{A}' = |\vec{A}'||\vec{\Omega}|dt\ \hat{n} \quad ,$$

where $|\vec{A}'| = |\vec{A}|\sin y$ (see above), $|\vec{\Omega}|dt$ is the angle of rotation **a**; $\hat{n}$ is the unit vector perpendicular to $\vec{A}'$.

Therefore

$$d\vec{A}' = |\vec{A}|\sin y\ |\vec{\Omega}|dt\ \hat{n}$$

or

$$d\vec{A}' = \vec{\Omega} \times \vec{A}.$$

Thus the general result for the time rate of change for any vector $\vec{A}$ is

$$\left(\frac{d\vec{A}}{dt}\right)_{fixed} = \vec{\Omega} \times \vec{A} + \left(\frac{d\vec{A}}{dt}\right)_{rot} = \left[\left(\frac{d}{dt}\right)_{rot} + \vec{\Omega} \times \right]\vec{A}$$

Applying the above rule (i.e. operator) twice to the position vector $\vec{r}$ of a particle (or water parcel) gives the following for the particle acceleration in the inertial or fixed frame of reference:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{fixed} = \left[\left(\frac{d}{dt}\right)_{rot} + \vec{\Omega}\times\right]\left[\left(\frac{d\bar{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2\Omega\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{\Omega}\times\vec{r}\,.$$

Since Newton's 2nd Law for the motion of a particle of mass **m** in an inertial is

$$\frac{\sum\vec{F}_{ext}}{m} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{fixed}$$

and in a rotating system is

$$\frac{\sum\vec{F}_{rot}}{m} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot}\,.$$

the above allows us to write

$$\sum\vec{F}_{ext} = \sum\vec{F}_{rot} + m(2\vec{\Omega}\times\vec{v}_{rot}) + m(\vec{\Omega}\times\vec{\Omega}\times\vec{r})$$

Defining the pseudo-forces in the rotating frame of reference

$$\vec{F}_{Coriolis} = -\ m(2\vec{\Omega}\times\vec{v}_{rot}) \qquad \vec{F}_{centrifugal} = -m(\vec{\Omega}\times\vec{\Omega}\times\vec{r})$$

allow us to write the following

$$\sum\vec{F}_{rot} = \sum\vec{F}_{ext} + \vec{F}_{Coriolis} + \vec{F}_{centrifugal}\,.$$

The Coriolis Force in the Ocean.

For a fluid parcel, the Coriolis force per unit volume is $-\ r\,(2\vec{\Omega}\ \mathrm{x}\ \vec{\mathrm{V}})$, where $\vec{\mathrm{V}}$ is the water parcel velocity, as seen by an Earth observer. Given the component forms of $\vec{\Omega}$ and $\vec{\mathrm{V}}$ in the local Cartesian coordinate system as shown in Figure B3

$$\vec{\Omega} = \Omega_h \vec{j} + \Omega_v \vec{k}$$

$$\vec{\Omega} = \Omega \cos f \vec{j} + \Omega \sin f \vec{k}$$

and

$$\vec{V} = u\dot{i} + v\vec{j} + w\vec{k} \qquad ,$$

the vector form of the full Coriolis force can be written

$$-\ r\left(2\vec{\Omega}x\vec{V}\right) = r\left(fv - f\cot f w\right)\vec{i} + r\left(-\ fu\right)\vec{j} + r\left(f\cot f u\right)\vec{k} \qquad ,$$

.

where the Coriolis parameter $f = 2\Omega \sin f$.

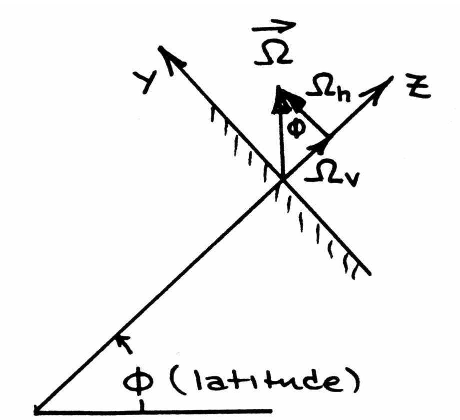


Figure B3. The components Earth rotation rate $\vec{\Omega}$ in a local Cartesian coordinate system at latitude f .