# CHAPTER 5
# ELEMENTS OF DYNAMICAL OCEANOGRAPHY

## Introduction

In the previous chapter, descriptive pictures of the distribution of ocean water temperature and salinity were used to infer the "mean" ocean circulation. Matthew Maury used ship logs to infer the principal global ocean current patterns, like what are shown in Figure 5.1a and Figure 5.1b. The complicated ocean current patterns are caused by different combination of forces acting in the different regions of the world's oceans. In order to make sense out of the apparent chaos, we must investigate a suite of simplified processes that occur in the ocean in terms of their kinematics and dynamics; the latter emphasizing the forces cause particular types of flows.

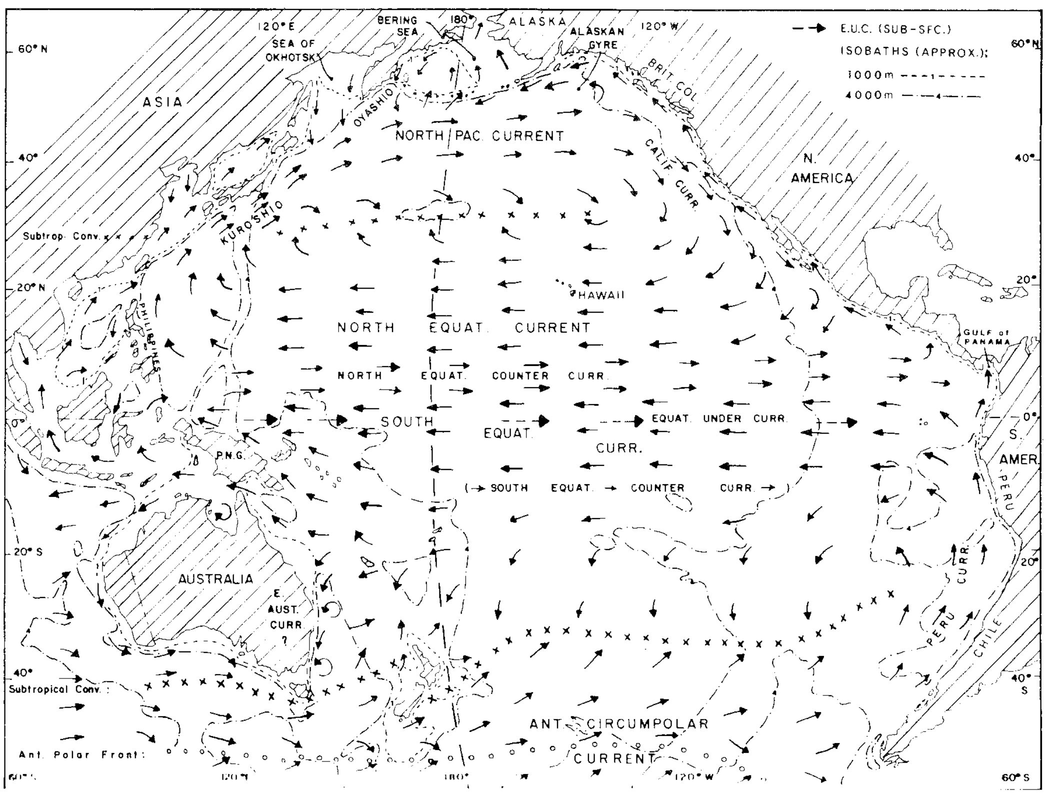


**Figure 5.1a** Pacific Ocean – surface circulation. (Pickard & Emery, 1982)

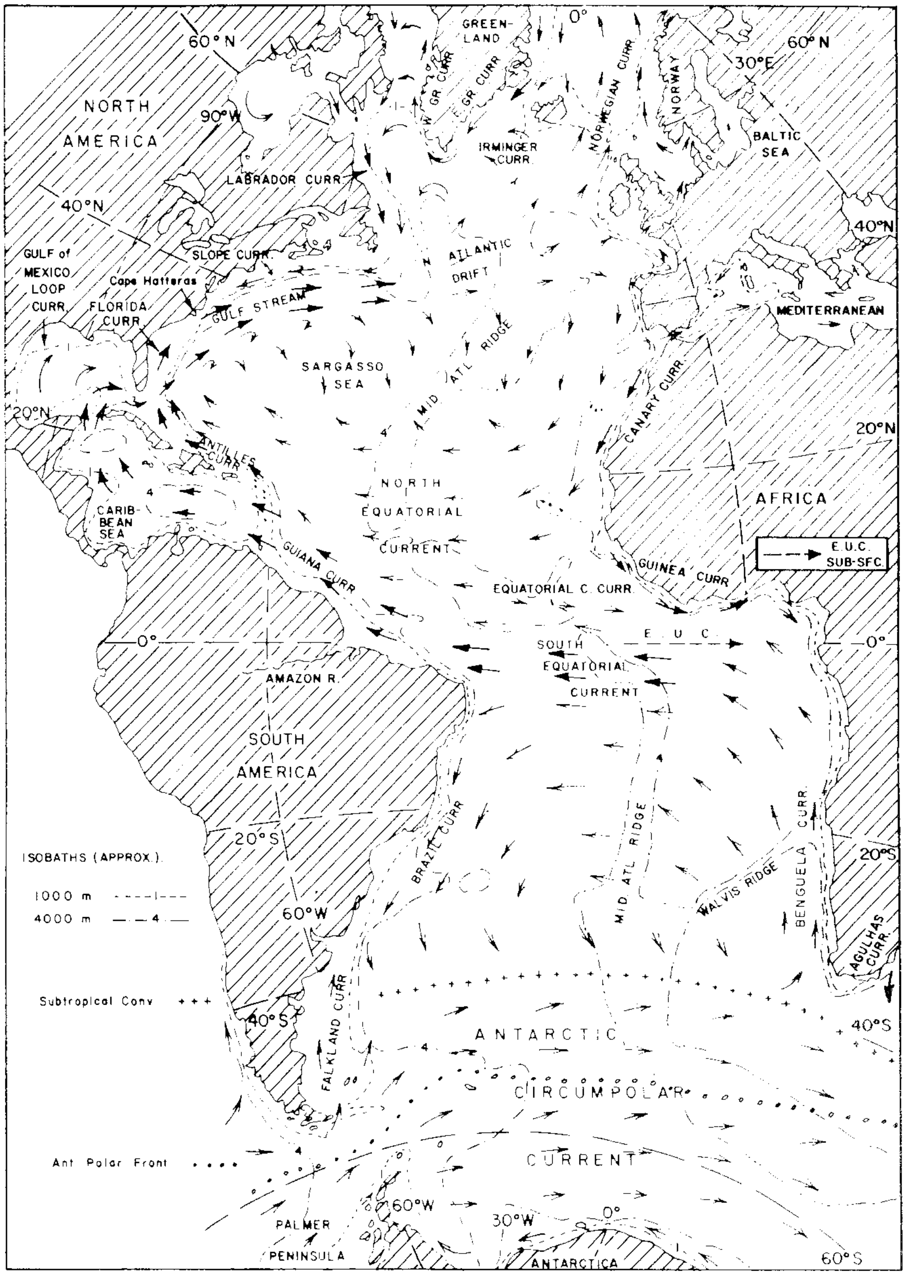


**Figure 5.1b** Atlantic Ocean – surface circulation. (Pickard & Emery, 1982)

## *Oceanic Variability*

The oceanic "mean" flow pictures above are deceptive because the ocean is highly variable changing on *time scales* ranging from seconds to millennia and *space scales* from mm to the 10000 km size of the ocean basin. Fortunately there is some order to it all. For example, the wave energy in a long-term, time series measurement record sea level from almost anywhere in the world's ocean would have a "periodogram" (Figure 5.3). That presentation shows that wave-related sea level fluctuations range in period (time between wave crests) from less than 1 second (for capillary waves) to periods greater than the energetic 12 hourly and 24 hourly tides.

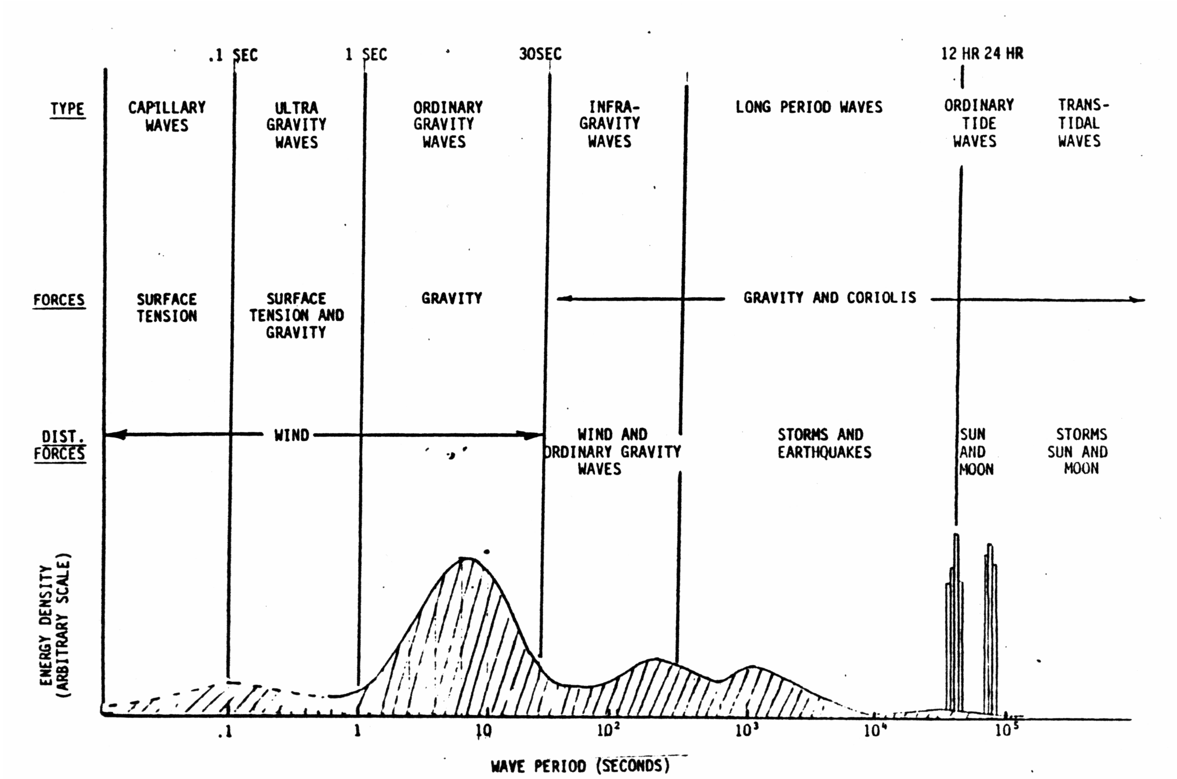


**Figure 5.3** Surface gravity wave periodogram – the distribution of relative energy (related to wave height squared) with respect to wave period.

The *time scales* of particular ocean processes (including waves) can be related to their particular *space scales* (e.g. wavelength for surface waves) in terms of a dispersion diagram as shown in Figure 5.4.

 

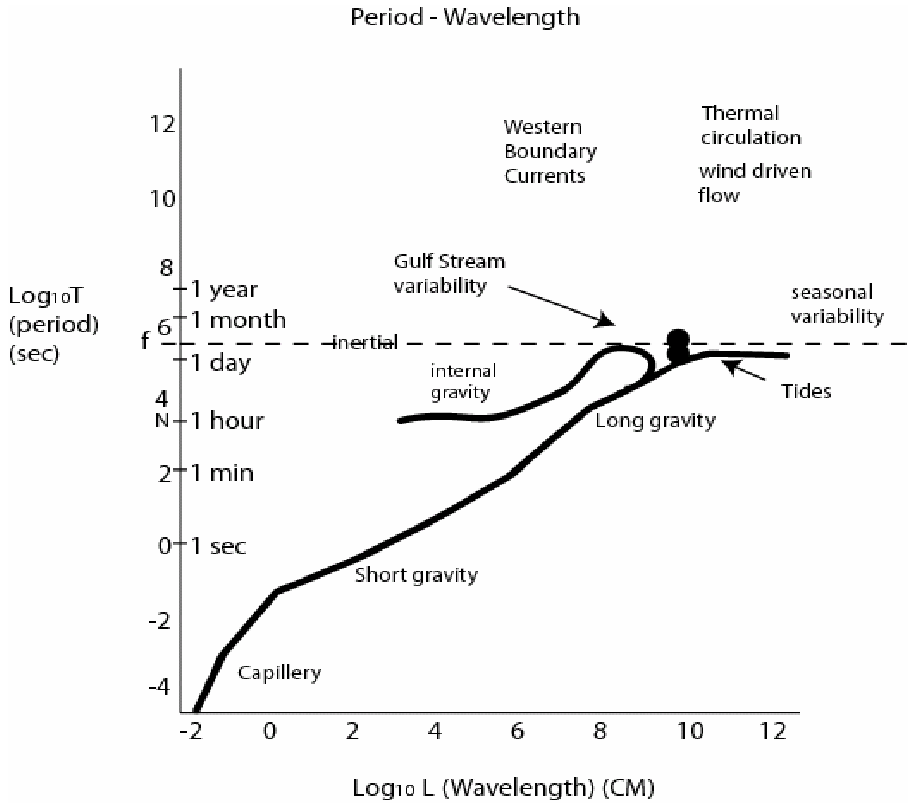


**Figure 5.4** A "dispersion diagram" relating the characteristic time scale T (in seconds) and space scales L (in centimeters) of important ocean phenomena.

## Profiles of Ocean Waves and Processes

Physics – Basic physical elements including forcing, restoring forces
Relevance

Capillary Waves
- P*hysics: surface tension / inertia / wind-driving*
- First effect of winds in wave generation

Surface Gravity Waves (short)
- *Physics: gravity / inertia / wind-driving*
- Mid-ocean wind waves/surface chop / shipping / off shore construction

Surface Gravity Waves (long)
- *Physics: gravity / inertia / bottom topography / wind-driving or earthquake*
- Surf and breakers / storm surges / tsunamis / dangerous flooding

Internal Gravity Waves
- *Physics: gravity or buoyancy / inertia / topography/ indirect wind-driving*
- Energy dissipation / mooring stresses / "dead water"

Tides
- *Physics: gravity / inertia / sun and moon gravitational pull / Coriolis force*
- Piloting and shipping / coastal construction

Internal Tides
- *Physics: gravity or buoyancy / inertia / topography/ indirect wind-driving*
- Internal waves at tidal periods with large vertical excursions

Planetary Waves
- *Physics: gravity / buoyancy / Coriolis force/ inertia*
- General ocean circulation / climate / weather

Sound Propagation
- *Physics: compressibility / inertia*
- Communications / echo sounding / biological measurements

Light Propagation
- *Physics: electromagnetic*
- Essential for biological productivity (no water motions)

Turbulence, Mixing
- *Physics: inertia / buoyancy / nonlinear interaction /, friction*
- Dissipation of energy/ mixing of momentum and dissolved chemicals inc. $O_2$

Estuary Flows
- *Physics: gravity-buoyancy/ Coriolis, tidal, and wind forcing*
- Pollutant dispersal, biological nutrient renewal, inflow of fresh water

Upwelling (and Downwelling)
- *Physics: buoyancy, local winds and currents*
- Brings up nutrients / formation of water masses

Mid-Ocean Mean Flows
- *Physics: buoyancy, Coriolis force, wind and heating driving, friction*
- Maintaining climate / distributing chemicals and heat for biology

Currents
- *Physics: buoyancy / Coriolis force / nonlinear inertia*
- Transfer of heat & vorticity / climate & local weather/ biology, chemistry

 

**Table 5.1 Characteristic** time and length scales associated with different ocean processes and currents.

| Type | Time Scales T | Length Scales L | Amp H |
|---|---|---|---|
| Mid Ocean "Mean" Flow | 1000 km | 1000 km | cm/sec |
| Western Boundary Currents | seasonal-decadal | 100 km | ~ 200 cm/sec |
| Upwelling/Downwelling | days-seasonal | 100 km | <1 mm/day |
| Continental Shelf Currents | days-months | 100 km | cm/sec |
| Estuarine Currents | day-seasonal | 100 km | cm/sec |
| Turbulence (Mixing) | sec | cm-m | mm-cm/sec |
| Capillary Waves | <0.1 sec | <1 cm | <1 cm |
| Short Surface Gravity Waves | 0.1s - 60s | cm-km | m |
| Long Surface Gravity Waves | sec/hr | m-10's km | m |
| Internal Gravity Waves | min-hrs | m-10's km | m |
| Tides | day | 1000 km | m |
| Internal Tides | day | 10-100 km | 10m |

 

The relative importance of these different ocean phenomena to the overall ocean variability can be inferred from their energy spectra – i.e. the distribution of energy versus period of variability. The form of an energy spectrum in Figure 5.4 depicts total energy in a particular band of periods T (i.e. process) in terms of area under the curve. The importance of capillary waves to overall surface wave is indicated here. The spectrum of density of energy within a fixed band of periods – i.e. energy density – is an alternate way to display energy spectra. After we have developed the concepts of oceanic energy transport and discussed the physical mechanisms associated with many of the most important ocean phenomena, we will come back to these pictures. A general discussion of energy follows next.

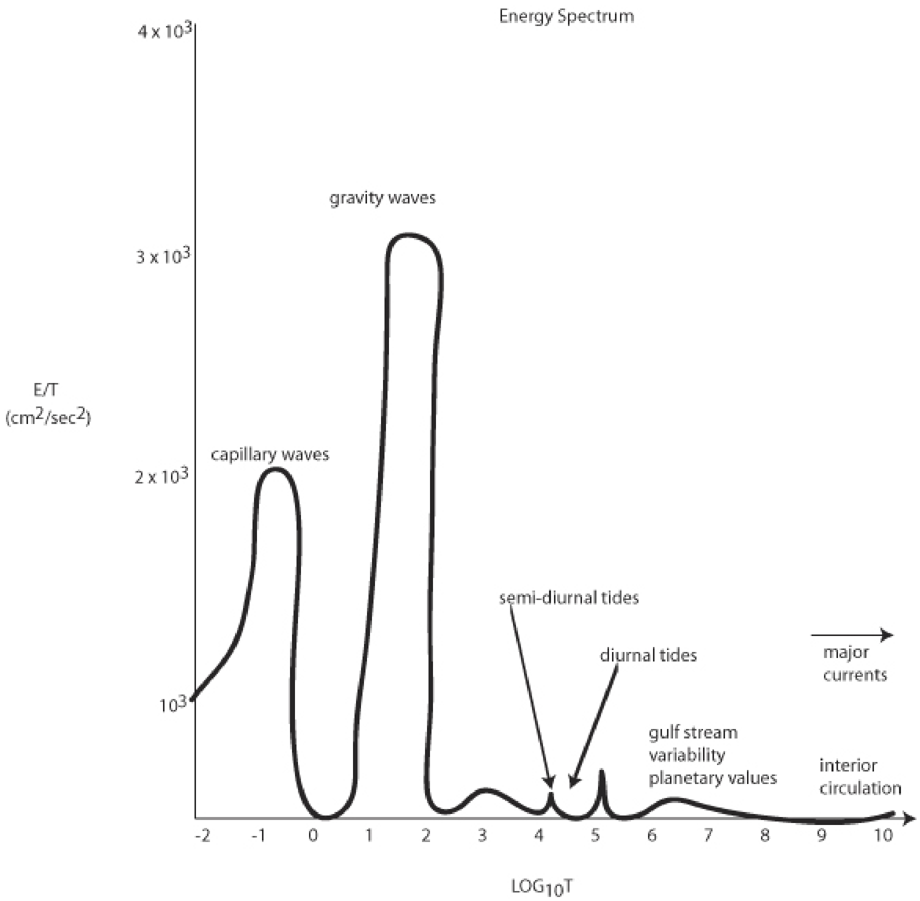


**Figure 5.4** Variance-preserving form of an energy spectrum of the different ocean processes.

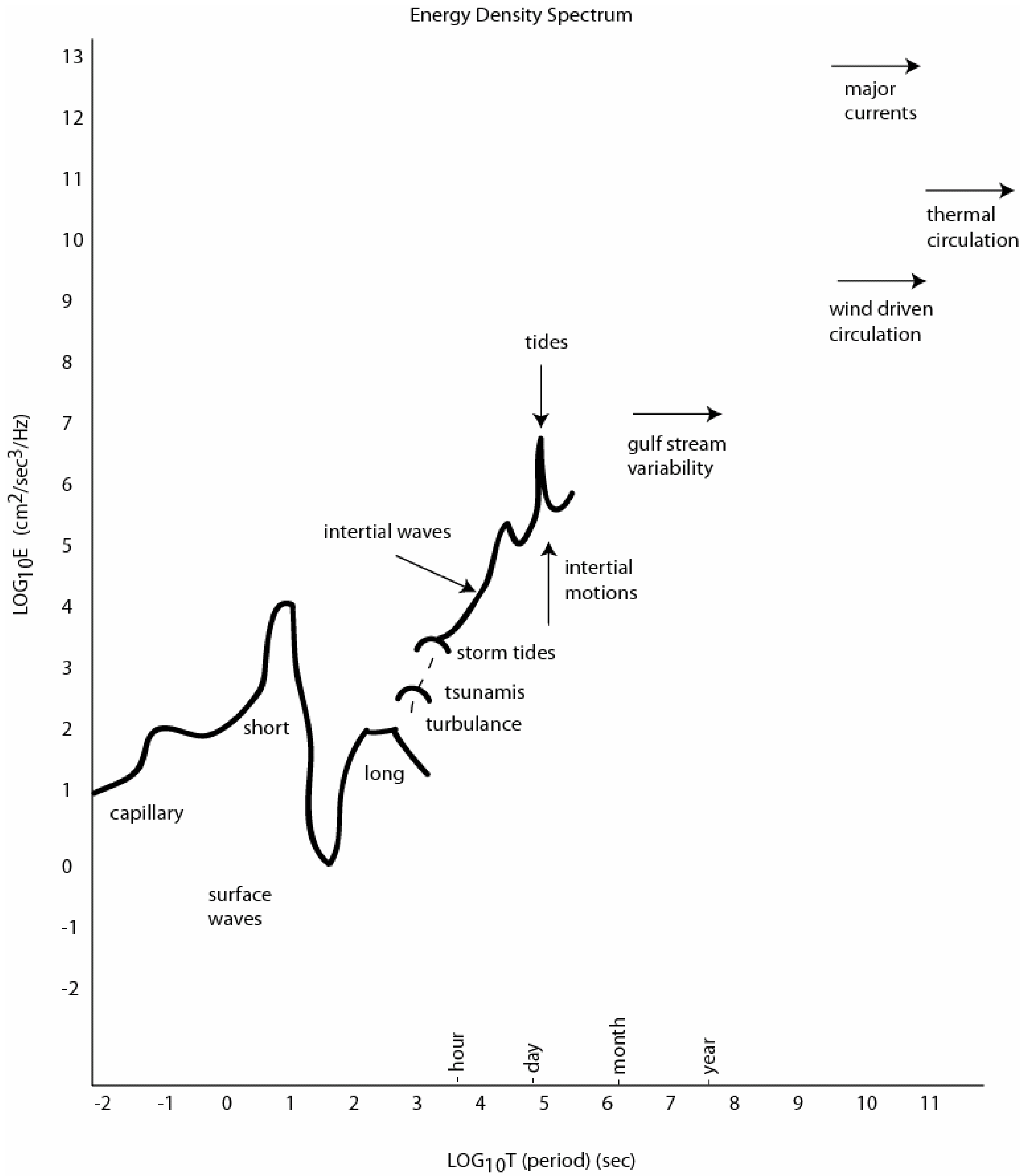


**Figure 5.5** Energy density spectrum for different oceanic motions

## *Forms of Energy*

Energy comes in different forms including mechanical, thermal and chemical to name a few. Mechanical energy can be divided into kinetic and potential energy forms.

**Kinetic energy** (or *energy of motion*) of a particle with mass m is

$$K = 1/2\, m\, (\vec{V} \bullet \vec{V}) = 1/2 m V^2 \quad ,$$

where $\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$ .

The time rate change of the kinetic energy of a particle can be shown to be equal to the work rate of the forces acting upon it by considering that the above is

$$\frac{dK}{dt} = \frac{1}{2} m\, (\vec{V} \bullet \frac{d\vec{V}}{dt} + \frac{d\vec{V}}{dt} \bullet \vec{V}) \qquad ,$$

which can be rewritten as

$$\frac{d\,K}{dt} = m\, \frac{d\vec{V}}{dt} \bullet \vec{V} = \vec{F} \bullet \frac{d\vec{r}}{dt} \, ,$$

work rate

where $\vec{F}$ is the force (or vector sum of forces) acting on the particle and $\vec{r}$ is the vector displacement of the particle. In other words, to change the kinetic energy of a particle, forces must do work (i.e. move) the particle. The amount of change in K equals the sum of the <u>work rates</u> in the three orthogonal directions or

$$\frac{dK}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \qquad .$$

For a **fluid domain,** like the ocean, we must keep track of the kinetic energy for each *fluid parcel* that makes up the domain. (Remember that a fluid parcel has a volume that is small enough so that its properties are homogeneous, yet large enough so that individual water molecules are indistinguishable). If the kinetic energy per unit volume of the *ith* fluid parcel is

$$1/2\, r_{\mathrm{i}}\ \Delta v_{\mathrm{i}} \left(\vec{V}_i \bullet \vec{V}_i\right),$$

where $r_{\mathrm{i}}\ \Delta v_{\mathrm{i}}$ is the product of the density and volume respectively of the ith parcel then the total kinetic energy of the whole system of parcels is

$$K = \sum_i \ 1/2\, r_{\mathrm{i}}\ \Delta v_{\mathrm{i}} \left(\vec{V}_i \bullet \vec{V}_i\right).$$

In the limit as the volume of the i$^{\mathrm{th}}$ parcel $\Delta \mathrm{v_i} \rightarrow 0$ , the sum above becomes the following integral

$$\mathrm{K} = \frac{1}{2} \int \mathrm{dx} \int \mathrm{dy} \int \mathrm{dz}\ r(\mathrm{x, y, z})\ \mathrm{V}^2(\mathrm{x, y, z}) \qquad ;$$

Further assuming that water density is 1 gm/cm$^3$, then the above can be written

$$\mathrm{K} = \frac{1}{2} \int \mathrm{dx} \int \mathrm{dy} \int \mathrm{dz}\ \mathrm{V}^2(x, y, z),$$

where $\frac{1}{2}\, r\, \mathrm{V}^2; units \left[\frac{\mathrm{gm\ cm^2/sec^2}}{\mathrm{cm^3}}\right]$; or $\frac{1}{2} \mathrm{V}^2$; units $\left[\mathrm{cm^2/sec^2}\right]$

is the kinetic <u>energy per unit volume</u> or <u>energy density</u> at the location x, y, z.

**Potential Energy** (or *energy of position*) is the other form of mechanical energy. Changes in potential energy are also related to work done. To see this consider the case of a *conservative force* (i.e. non-dissipative force, like gravity) acting on a particle, with a mass m, as it moves along any of the paths from point A to B as shown in Figure 5.6.

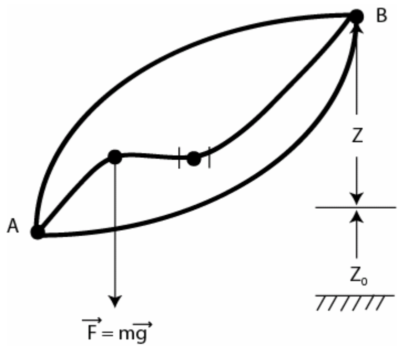


**Figure 5.6** The change in potential energy of a mass m that is moved in differential increments of $d\vec{s}$ along a path from A to B in a gravitational field.

The potential energy U, relative to a reference elevation $z_o$, is the work done on the particle (by the vector sum of the forces - gravity in this case) according to

$$\text{Potential Energy} \equiv U \equiv \int_A^B \vec{F} \bullet d\vec{s} = |\vec{F}|(z - z_o) = mg(z - z_o) \quad ,$$

where $d\vec{s}$ is a unit vector tangent to the local path and $\vec{g}$ is the acceleration due to effective gravity. (Note that only the vertical displacement is relevant in the case of gravity).

For a of a fluid ocean domain, the total potential energy at a particular time can, like that for total ocean domain kinetic energy, be written as

$$U = \sum_i r_i \, \Delta v_i g\left(z_i - z_o\right)$$

or more accurately

$$U = \int dx \int dy \int dz \; r(x, y, z) \, g(z - z_o) \quad .$$

where $r(x, y, z)$ is the instantaneous ocean density field .In such a conservative system, potential energy can be converted to kinetic energy reversibly (i.e., without energy loss), according to K+U = constant. The fluid ocean is nearly conservative, because dissipation is relatively small.

However, not all of the total potential energy of the oceanic system is available for conversion to kinetic energy. For example, imagine a reference ocean in which all of its isopycnals are level (i.e. they lie parallel to geopotential surfaces; see Figure 5.7). In such a reference ocean, with $r^R(\mathrm{x, y, z})$ , none of the potential energy can be converted to kinetic energy. The only available potential energy (or AU) in the ocean is that associated with isopycnals that are disturbed relative to the above reference state.

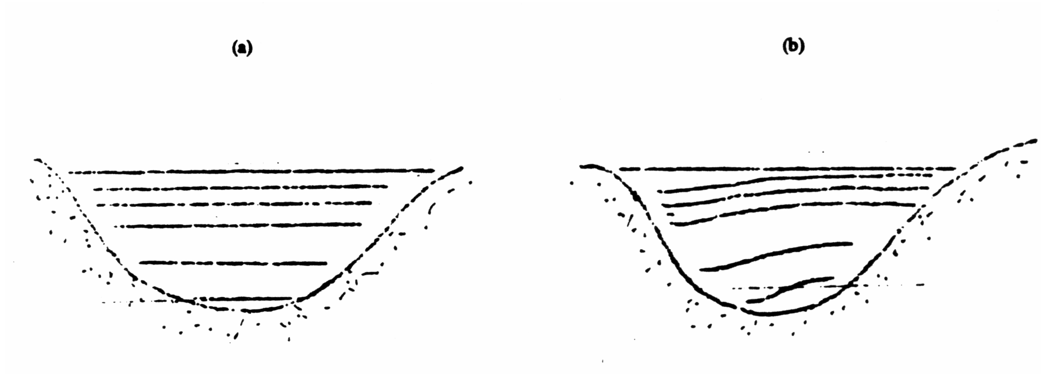


**Figure 5.7.** (a) Undisturbed reference density field $r^R(\mathrm{x, y, z})$ composed of isopycnals coincide with geopotential surfaces and thus are level by definition; (b) Disturbed density field composed of isopycnals that are not "level".

The finite difference form of the domain total available potential energy is

$$AU = \sum_i r'_{\mathrm{i}} \, \Delta v_{\mathrm{i}} g\left(z_i - z_o\right) \quad ,$$

where $r'(\mathrm{x, y, z}) = r(\mathrm{x, y, z}) - r^R(\mathrm{x, y, z})$ is a density anomaly field. The more accurate form is

$$A\mathrm{U} = \int \mathrm{dx} \int \mathrm{dy} \int \mathrm{dz}\; r'(\mathrm{x, y, z})\, \mathrm{g}(\mathrm{z} - \mathrm{z_o})$$

Thus the more relevant conservation of mechanical energy statement is

$\mathrm{K} + A\mathrm{U} = \text{constant}$ . The swinging of a pendulum is a good example of the continuous conversion of potential to kinetic energy and visa versa.

 

## *The Application of Newtonian Dynamics to a Fluid Ocean*

Here we apply Newton's laws to a continuum – the fluid ocean - instead of solid objects. The first step in formulating the problem is to assume that the ocean can be subdivided into fluid parcels. Newton's Second Law of Motion for a fluid relates the vector sum of all the relevant forces on a fluid parcel to the resulting time-rate change of momentum of the fluid parcel. This is a statement of the conservation of momentum for the fluid parcel. Since different fluid parcels cannot occupy the same space simultaneously, we must solve Newton's 2 $^{nd}$ Law subject to the constraint of the conservation of mass (or volume). For some problems, the solution is also subject to an explicit constraint of the conservation of energy, including heat and mechanical energy.

The problem can be formulated mathematically once we choose a coordinate system; generally the Cartesian coordinate system shown in Figure 5.8. This coordinate system is generally embedded in the Earth such that the x-axis points eastward, the y-axis northward, and local z-axis upward; and by convention the u, v, and w velocity components point is the x, y and z directions respectively.

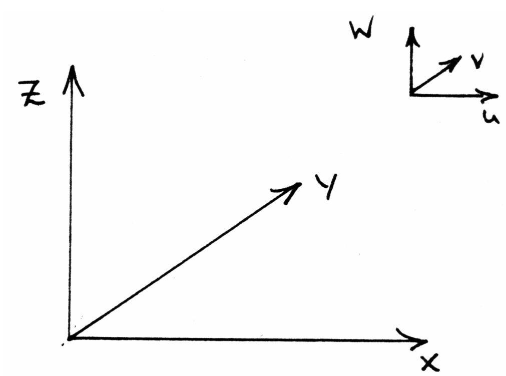


**Figure 5.8.** The Cartesian coordinate system is composed of the mutually perpendicular x- (eastward), y- (northward) and z- (local upward) axes; By convention the u, v and w velocity components are aligned with the x, y and z axes respectively.

We generally seek to solve the problem of interest at all locations in the ocean domain, in terms for the time histories of the relevant variables of the problem usually including:

 

- the velocity components

$$u = u(x,y,z,t)$$
$$v = v(x,y,z,t)$$
$$w = w(x,y,z,t)$$

- the pressure and density fields

$$p = p(x,y,z,t)$$
$$r = r\ (x,y,z,t)$$

- and the temperature and salinity fields

$$T = T(x,y,z,t)$$
$$S = S(x,y,z,t)$$

which of course are related to density through the equation of state.

Valid solution(s) to the stated problem means that they simultaneously satisfy the following

**Conservation of Mass**

**Conservation of Momentum**

**Conservation of Energy** (if needed)

as well as the

4) **Boundary and Initial Conditions**

What are the **conservation of mass** and **momentum equations** for a fluid? To begin to answer the question, we must choose one of the two approaches to represent the problem. *One approach* is to use the Lagrangian representation, in which the motion of each fluid parcel (identified by its initial position) is described as a function of time. However, this approach is complicated and used only is special circumstances. The *other approach* is to use the Eulerian representation, in which the time evolution fluid motion and properties are described at all fixed points within a defined domain. In this course, *we will normally use the Eulerian representation*.

**Conservation of Mass**

In deriving the conservation of mass (or volume) statement, we will assume a steady state system – i.e., no changes with time. First, determine the amount of mass transport through a very small control volume (with dimensions dx, dy, and dz) in a short time dt, as shown in Figure 5.9.

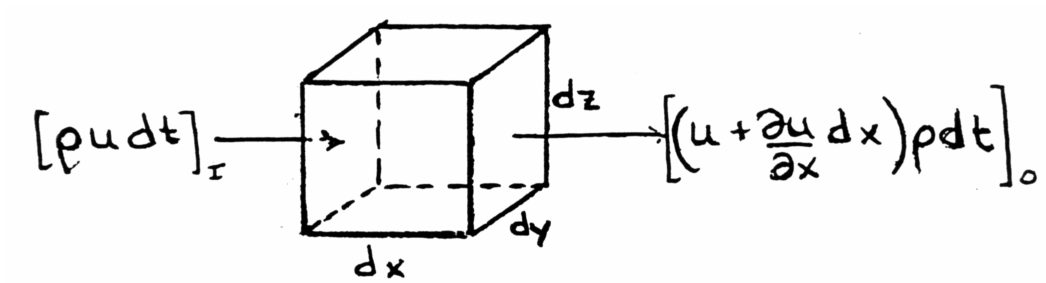


**Figure 5.9.** A schematic of the x-directed mass transport through a small control volume in a small time dt.

The <u>net mass transport</u> in the x direction in time dt is

$$r[(u + du) - u]\, dy\, dz\, dt \qquad ,$$

where $du = \frac{\partial u}{\partial x} dx$ is a small increment in the horizontal velocity u.

Thus the net x-directed mass becomes

$$r[(u + \frac{\partial u}{\partial x} dx) - u]\, dy\, dz\, dt$$

or

$$\frac{\partial u}{\partial x}\, r\, dx\, dy\, dz\, dt$$

Likewise the net y- directed *mass* is

$$\frac{\partial v}{\partial y}\, r\, dx\, dy\, dz\, dt$$

and the net z- directed *mass* is

$$\frac{\partial w}{\partial z}\, r\, dx\, dy\, dx\, dt.$$

If we can assume that:

(a) $r$ is constant;

(b) fluid is incompressible; and

(c) there are no sources or sinks within the domain, then the <u>sum of the net transports</u> is

$$r \ \mathrm{dx\ dy\ dz\ dt} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \equiv 0 \qquad ;$$

from which the *differential form* of the Continuity Equation follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad .$$

Using the "del operator", defined as

$$\nabla \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

we can derive the vector form of the Continuity Equation

$$\nabla \bullet \vec{V} = 0$$

which is the called divergence of $\vec{V}$ .

The above forms of the continuity *finite difference form* of the continuity equation is

$$\frac{d\,u}{d\,x} + \frac{d\,v}{d\,y} + \frac{d\,w}{d\,z} \approx 0.$$

The implications of the continuity relation for a fluid can be explored in terms of a fluid parcel as it passes through a region in x, y, z space. You are a scuba diver looking upward at a cubical dyed water parcel at the surface as it moves from left to right along the y-axis in Figure 5.10. You observe the parcel shortening in the x-direction, *but not in the y-direction* as it moves through the control volume toward + y. Thus $\frac{d\,u}{d\,x}$ for the parcel is **negative.** The continuity relation requires that this x-directed shortening be compensated by a positive or $+\left[\frac{dv}{dy} + \frac{dw}{dz}\right]$. Since $\frac{dv}{dy} = 0$, $\frac{dw}{dz}$ must be positive. The observed downward stretching or $\frac{(-dw)}{(-dz)}$ is consistent with this conclusion.

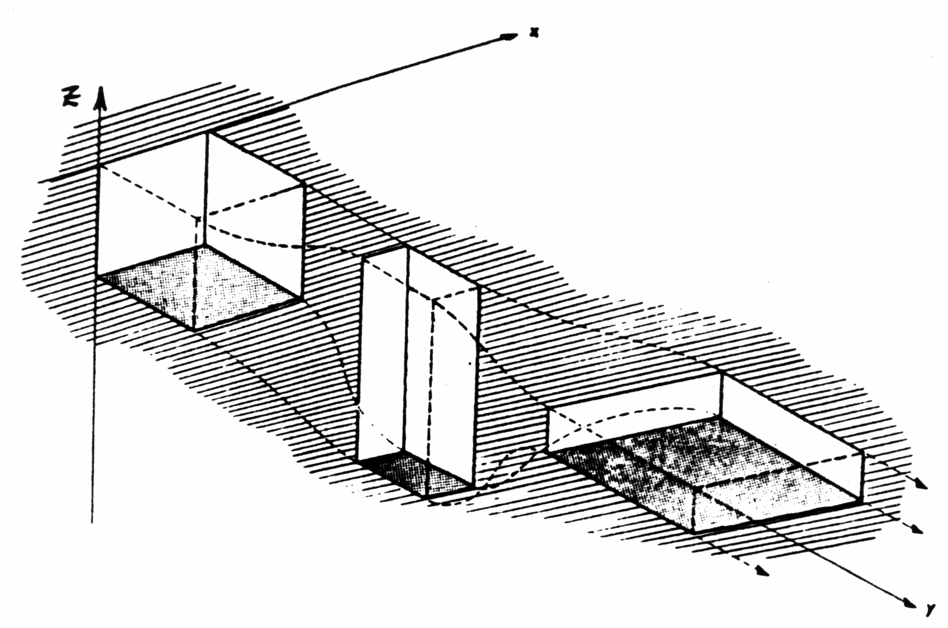


**Figure 5.10.** Conservation of volume as seen from *below the sea surface*. (Von Arx, 1974)

## Conservation of Momentum

For a fluid, Newton's 2nd Law per unit volume is:

$$\sum \vec{F} = \frac{d}{dt}\, r\, \vec{V}$$

If the density of a particular fluid parcel is constant then we can write

$$\sum \vec{F} = r\, \frac{d\vec{V}}{dt},$$

where $\vec{V} = u\,\vec{i} + v\,\vec{j} + w\vec{k}$ is the total velocity vector and $\sum \vec{F}$ represents the vector sum of pressure, gravity, friction and even the Coriolis forces on the fluid parcel.

 

What is $\frac{d\vec{V}}{dt}$ for a fluid?

From an Eulerian point of view $\frac{d\vec{V}}{dt}$ is the acceleration of a fluid parcel as it moves through a control volume at the point of interest with coordinates x, y, z. If the x velocity is u = u(x,y,z,t) then a small change in u (i.e. du) experienced by the parcel can be expressed

$$du = \frac{\partial u}{\partial t}\Big|_{x,y,z} dt + \frac{\partial u}{\partial x}\Big|_{y,z,t} dx + \frac{\partial u}{\partial y}\Big|_{x,z,t} dy + \frac{\partial u}{\partial z}\Big|_{x,y,t} dz\,.$$

If du occurs in a time dt, then the x-directed (or eastward) component of the acceleration is

$$\frac{du}{dt} = \underset{(a)}{\frac{\partial u}{\partial t}} + u\underset{(b)}{\frac{\partial u}{\partial x}} + v\underset{(c)}{\frac{\partial u}{\partial y}} + w\underset{(d)}{\frac{\partial u}{\partial z}}.$$

The <u>substantial derivative</u> of u or $\frac{du}{dt}$ is composed of the (a) local acceleration (i.e. the <u>local</u> time rate change of u) <u>and</u> convective accelerations, resulting from fluid parcels being swept by the respective component velocities through a flow field with spatial gradients of u in the respective directions. In order to get a better feeling for these convective acceleration terms, consider what happens when the local v velocity component sweeps a fluid parcel through a flow field with a positive northward gradient of the u velocity component - u current shear $\frac{\partial u}{\partial y}$ (Figure 5.11).

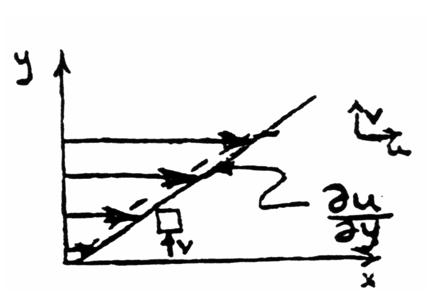


**Figure 5.11.** The x-directed convective acceleration arising from y-directed motion (v) of a parcel in a shear flow.

 

In this case the parcel undergoes an x-directed acceleration because it is being displaced northward from slower eastward (or u) flow into a region of faster eastward flow. This component of the acceleration is due to the coupling; $v\frac{\partial u}{\partial y}$ .

The corresponding y- and z-directed components of acceleration are

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}$$

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}$$

Since $$\frac{d\vec{V}}{dt}=\frac{du}{dt}\vec{i}+\frac{dv}{dt}\vec{j}+\frac{dw}{dt}\vec{k},$$

we can write the substantial derivative of $\vec{V}$ in the following vector form

$$\frac{d\vec{V}}{dt}=\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\bullet\nabla\right)\vec{V}$$

**For most of our purposes** the convective acceleration contributions are approximately zero. Thus $$\frac{d\vec{V}}{dt}\approx\frac{\partial\vec{V}}{\partial t}$$

 

## Forces

What does $\sum \vec{F}$ mean for a fluid?

In this course we are principally concerned with pressure gradient forces, gravitational forces, frictional forces and forces related to rotation such as the Coriolis force and centrifugal force.

The **pressure gradient force** (per unit volume) is a body force which acts on a fluid volume. Consider the pressure force balance on the fluid control volume in Figure 5.12. Remember that pressure always acts normal (or perpendicular) to the face of the control volume and has units of force per unit area or ($MLT^{-2}/L^2$). In this case, the pressure is increasing from left to right.

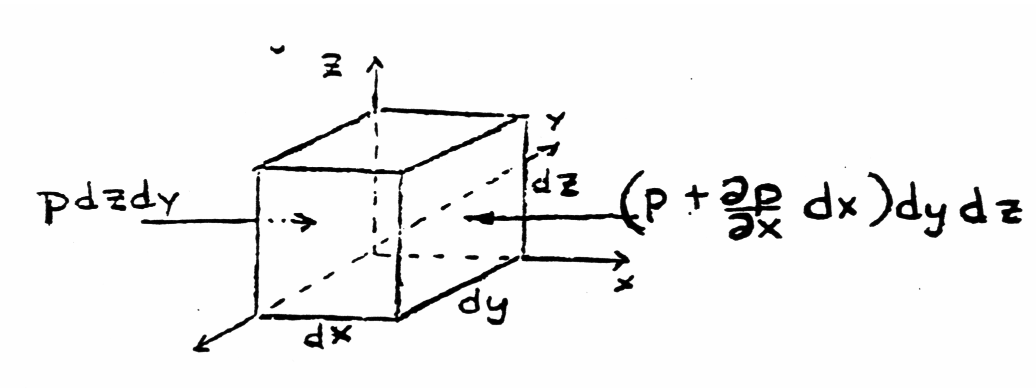


**Figure 5.12.** X-directed pressure forces on a fluid parcel.

The net x-directed pressure force on the control volume is

$$\sum F_x^p = [p - (p + \frac{\partial p}{\partial x} dx)]\, dy\, dz$$

$$= -\frac{\partial p}{\partial x} dx\, dy\, dz$$

The net x-directed pressure force (per unit volume) is

$$PGF_x = \sum F_x^p / dx\ dy\ dz = -\frac{\partial p}{\partial x} \qquad ,$$

and is called the x-directed pressure gradient force (or $PGF_x$). Note that $PGF_x$ is the *negative* x-directed pressure gradient. A positive pressure gradient has higher pressures at increasing x (to the right), lower pressures at decreasing x (to the left), and under these circumstances, tends to force water to the left.

When the y and z components of the pressure force are derived in a similar manner, we are able to find the total pressure gradient force per unit volume;

$$PGF = -\nabla p = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right) \qquad .$$

The *finite difference form* of the pressure gradient force

$$PGF_{fd} = -\left(\frac{dp}{d\,x}\vec{i} + \frac{dp}{d\,y}\vec{j} + \frac{dp}{d\,z}\vec{k}\right),$$

where a 2-station estimate of the x-pressure gradient is

$$\frac{dp}{d\,x} = p_2 - p_1 / x_2 - x_1.$$

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Now let's explore Newton's 2nd Law in the presence of differing combinations of the pressure gradient and gravitational forces.

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**(1) Assumption:** The pressure gradient force is the only force acting on the water parcels at a chosen Eulerian location.

For this case, Newton's 2nd law is

$$\rho\frac{d\vec{V}}{dt} = PGF = -\nabla p = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)$$

Thus when acting alone, pressure gradients accelerate fluid parcels from high pressure toward low pressure zones. The only problem with this "model" is that there is nothing to keep the fluid from accelerating forever and the model is unrealistic! Thus we need to consider opposing forces such as the *gravitational force*.

The **gravitational force** on a water parcel (or weight) per unit volume

$$\vec{F}^{g} = - r\, g\vec{k} \qquad .$$

is a body force on a fluid parcel; one that is proportional to mass and acts only downward.

**(2) Assumption:** The pressure gradient and gravitational forces are the only two forces acting on the fluid parcel in question. Under these circumstances, Newton's 2nd law becomes

$$\sum \vec{F} = r\,\frac{d\vec{V}}{dt} = [\underbrace{-\nabla p = -(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k})}_{\text{PGF}}] \underbrace{-\ r\, g\vec{k}}_{\text{gravitational}}$$

Now **assume** that there is no motion (i.e. $\vec{V} \equiv 0$). Thus $d\vec{V}/dt = 0$ and we have only a balance between the pressure gradient and gravity forces – a *force balance-* in the vertical direction *only*; according to

$$-\frac{\partial p}{\partial z} - rg = 0$$

or

$$\frac{\partial p}{\partial z} = -rg \qquad .$$

Thus this static situation is maintained by the exact balance between the upward local pressure gradient and the downward weight of the water parcel, which of course is motionless!

These *hydrostatic conditions* are explored further by integrating the above as follows.

First, since p = p(z) *only*, the partial derivative $\frac{\partial}{\partial z}$ becomes the total derivative $\frac{d}{dz}$ and the above can be written

$$\frac{dp}{dz} = -rg \ .$$

Integrating the above from the sea surface (at z = 0, where p = atmospheric = $p_a$) to an arbitrary elevation z is

$$\int_{p_a}^{p(z)} dp = -\int_0^z rg\, dz \qquad .$$

Evaluating the integral on the left hand side leads to

$$p(z) - p_a = -\int_0^z r(z)g \ \ dz \ .$$

Since density variations with depth $r(z)$ are relatively small, we can decompose it into

$$r(z) = r_0 + r^{/}(z)$$

where $r_0$ is the relatively large mean density and $r^{/}$ is the relatively small density anomaly (i.e., $r^{/} << r_0$).

Substituting the above this approximation into the previous equation and integrating, yields the array of contributions to the hydrostatic pressure p(z) at an arbitrary elevation

$$p(z) = \underset{(a)}{-r_0\, gz} - \int_0^Z \underset{(b)}{r^{/}}\, g(z)\ dz + \underset{(c)}{p_a} \ .$$

What are these components?

Term (a), usually the dominant contribution, is the weight of the water column above the selected depth (i.e., –z) in a constant density ocean;

Term (b) is the very small contribution to the total hydrostatic pressure, due to the integrated weight of a variable density anomaly;

Term (c) is the contribution due to atmospheric pressure;

Let's simplify as much as possible! First, Term (b) < < Term (a) virtually all the time. Since the average atmospheric pressure is about 1 atmosphere (which is ~ 1020 millibars = 10.2 decibars, where 1 bar $= 10^6$ dynes/cm$^2$ ), Term (c) < < Term (a), except for the shallowest depths. Therefore to a *reasonable degree of accuracy* the more familiar form

$$p(z) = -rgz$$

can be used for computing *hydrostatic pressure*.

For Example

$$p = -r\,gz$$

Assuming $\begin{array}{l} r = 1.025 gm/cm^{-3} \\ g = 0.98x10^3 cms^{-2} \end{array}$ ===> $r\,g = 1.0045x10^3 \dfrac{gm}{cm^2 - s^2}$ .

Thus at an elevation z = -1m (actually a depth of 1m):

$$p(z = -1\text{m}) \sim 10^3 \bullet 10^2 = 10^5 \frac{\text{gm cm s}^{-2}}{\text{cm}^2}$$

$$p(-1m) \sim 10^5 \frac{\text{dynes}}{\text{cm}^2} = 0.1 \text{ bars (decibar)}$$

---

Now let's compute the *internal pressure gradient field* in the following suite of model ocean situations.

**(A) Homogeneous Undisturbed Ocean**

Given the homogeneous ocean (i.e. $s_q$ = constant) with no sea surface tilt in Figure 5.13, what is the bottom pressure difference between stations **a** and **b,** which are separated laterally by a finite distance dx.

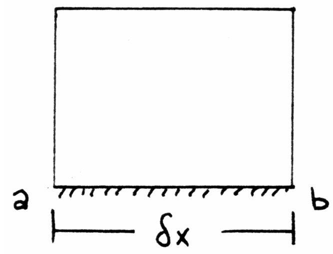



**Figure 5.13.** Pressure gradient force in a homogeneous ocean no sea surface tilt.

Since $p_b = p_a$ , $\quad d\ p = p_b - p_a = 0$ .

### (B) Two Dimensional Homogeneous Ocean with Surface Tilt:

Consider the uniform ocean (i.e. $s_q = r$ = constant) with surface tilt in Figure 5.14.

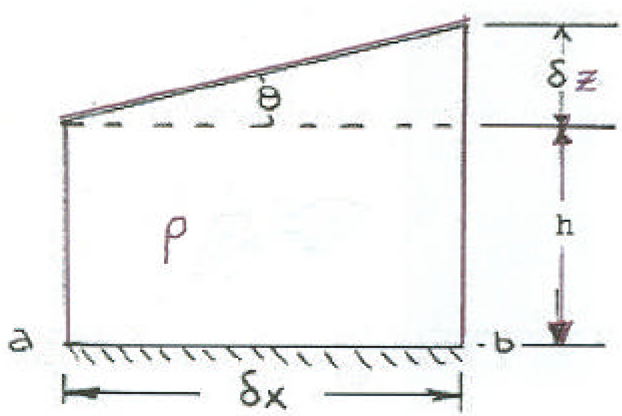



**Figure 5.14** Pressure gradient force in a homogeneous ocean with sea surface tilt.

The finite difference form of the bottom pressure gradient force is,

$$
\begin{aligned}
&= -\frac{dp}{dx} = -\frac{(p_b - p_a)}{dx} \\
PGF_x \quad &= -\frac{g}{dx}[r\,(dz + h) - rh] \\
&= -\frac{g r dz}{dx} = -gr \ \tan q
\end{aligned}
$$

Pressure gradient force acts uniformly throughout water column! Thus, assuming no lateral boundaries, the full ocean water column moves horizontally from right to left as if it were "downhill". No real surprise here.

### C. Two-Layered Ocean Without Surface Tilt

Consider a two layer ocean, with no tilt (Figure 5.15), in which a homogeneous upper layer 1 overlies a homogeneous lower layer 2, where $r_2 \geq r_1$ with $r$ = constant) with surface tilt in Figure 5.15.

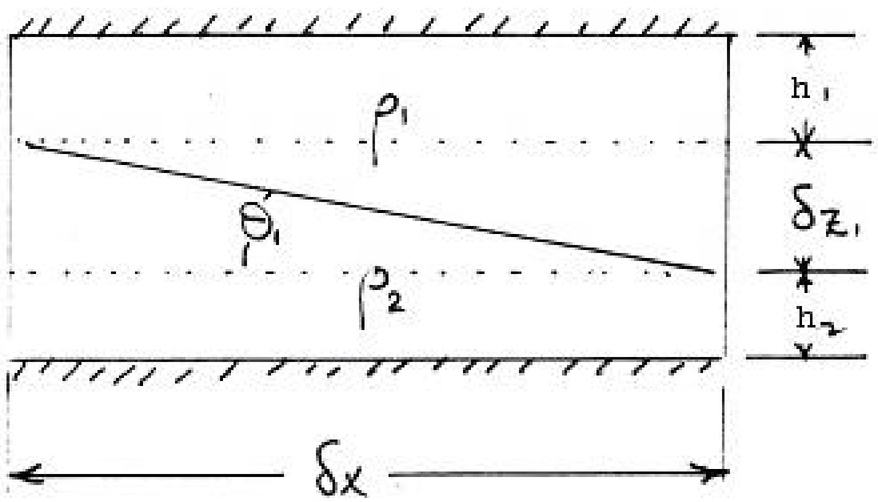



**Figure 5.15.** Pressure gradient force in a two layer model ocean with no sea surface tilt.

The pressure gradient force $PGF_x$ is

$$= -\frac{dp}{dx} = -\frac{(p_b - p_a)}{dx}$$

$$= -\frac{dp}{dx} = -\left\{\frac{g}{dx}\left[r_1(h_1 + dz_1) + r_2 h_2\right] - \frac{g}{dx}\left[r_1 h_1 + r_2(h_2 + dz_1)\right]\right\}$$

$$= -\frac{dp}{dx}\left[r_1 dz_1 - r_2 dz_1\right]$$

$$= \frac{g}{dx}(r_2 - r_1)dz_1 = g(r_2 - r_1)\tan q_1$$

At depths below the density interface, the horizontal pressure gradient force is uniform and in the direction from more dense to less dense water (i.e. from left to right).

*What is the horizontal pressure gradient above the interface?*

## *Friction Forces*

The friction force is a surface force that is most important near boundaries such as the surface, bottom and coasts. Friction leads to stress which like pressure has units of force/area but unlike pressure acts tangential to fluid volume surfaces.

We can understand the friction force better if we consider the x-directed stresses acting on the surface and bottom of a fluid control volume (Figure 5.16).

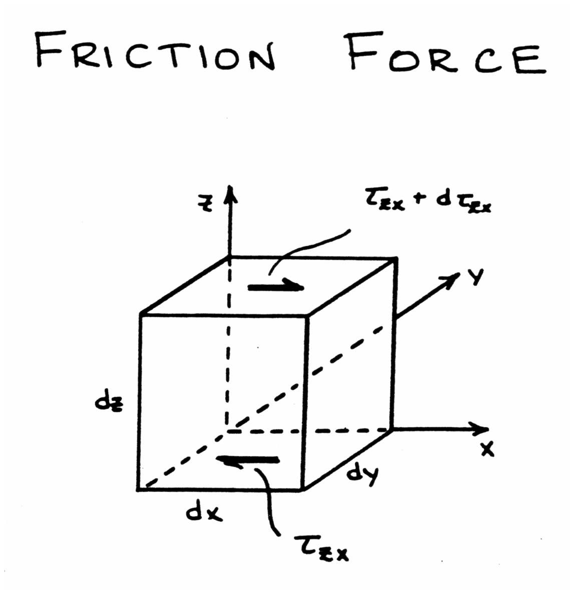



**Figure 5.16** The x-directed friction forces horizontal surfaces of a fluid parcel.

As you will see, the tangential friction-related stress is more complicated than the perpendicular compression- related pressure (i.e. 9 “tensor” components versus 3 vector components). To begin we will simplify our considerations by including only stresses acting on horizontal surfaces in the x direction only!

Following the approaches used above, we can write the

<u>net</u> x-directed friction force due to stresses on the horizontal surfaces $\Sigma F_x^h$ as

$$\Sigma F_x^h = \left[ t_{zx}(z) + \frac{\partial t_{zx}}{\partial z} dz - t_{zx}(z) \right] dx\, dy$$

or

$$\Sigma F_x^h = \frac{\partial t_{zx}}{\partial z} dz\, dx\, dy \quad ;$$

n<u>et</u> x-directed friction force due to stresses on the vertical surfaces $\Sigma F_x^v$ as

$$\Sigma F_x^v = \left[ t_{yx}(y) + \frac{\partial t_{yx}}{\partial y} dy - t_{yx}(y) \right] dx\, dz$$

or

$$\Sigma F_x^v = \frac{\partial t_{yx}}{\partial y} dy\, dx\, dz$$

Thus the **net x-directed friction force** (per unit volume) is

$$F_x^f = \frac{\partial t_{zx}}{\partial z} + \frac{\partial t_{yx}}{\partial y}$$

and arises because of :

(a) *Vertical Friction* - vertical gradients of the stress on horizontal surfaces;

(b) *Lateral Friction* - horizontal gradients of the stress on vertical surfaces

*What is the corresponding y- and z- directed friction forces?*

By analogy, the **net y-directed friction force** (per unit volume) is

$$F_y^f = \frac{\partial t_{zy}}{\partial z} + \frac{\partial t_{xy}}{\partial x}$$

and the **net z-directed friction force** (per unit volume) is

$$F_z^f = \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y}$$

## *Pseudo Forces*

An observer in a rotating frame of reference such as a turning car will feel a "centrifugal force" pushing them outward. An observer walking on a merry-go-round will feel a Coriolis force pushing them to the right of the direction in which they are walking. However, these examples are not true forces in that they are not observed as forces in an inertial frame of reference (i.e. a non-accelerating frame of reference). To an observer in an inertial frame of reference, what appear to be forces to an observer in an accelerating frame of reference, are really accelerations associated with the rotating (i.e. accelerating) frame of reference.

To gain a greater appreciation for these concepts consider a simpler situation. Consider the case in which Joe, who is standing on the train station platform, throws a briefcase to Moe, who has forgotten it and is already on the on the departing train. The questions are how do Joe and Moe observe the dynamics of the trajectory of the briefcase (i.e. the particle in Figure 5.17)?

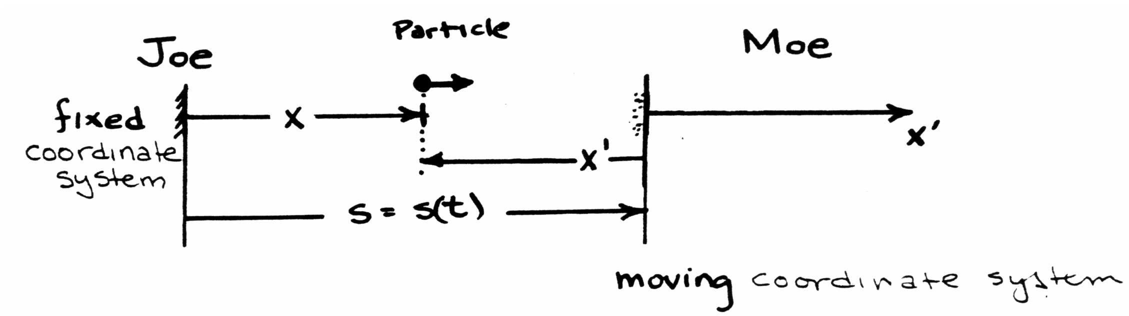



**Figure 5.17.** Particle motion as seen by a fixed (Joe) and moving (Moe) observer.

This is a one-dimensional situation in which is Joe is in a fixed frame of reference and measures horizontal positions in terms of x relative to the fixed platform. Moe is in a moving frame of reference and measures horizontal positions in terms of x' relative to the moving train. The position law relating the two coordinate systems is

$$x = x' + s\,,$$

where s is the distance from the train station to the moving train.

Given the position law, what is the acceleration of the particle (i.e. briefcase) as determined in both frames of reference?

Differentiating the position law twice gives

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} + \frac{d^2s}{dt^2} \; .$$

Thus, if either (1) s = constant or (2) s = ut for a constant velocity u, then acceleration is the same in both frames of reference.

However, if $s = \frac{at^2}{2}$ for a constant (i.e. Moe is accelerating relative to Joe), then

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} + a \; .$$

and the “observed “accelerations of the briefcase are different in the two systems. To determine the dynamical implications, apply Newton’s Second Law in the two systems. The total external forces on the briefcase in either system are $\sum F_x$ . Thus in Joe’s fixed system Newton’s Second Law is:

$$\sum F_x = m\frac{d^2x}{dt^2}$$

However, Newton’s 2nd Law in Moe’s system is:

$$\sum F_x - ma = m\frac{d^2x'}{dt^2} \; .$$

Thus in Moe’s system, the train’s times the mass of the briefcase is a pseudo-force (-ma) in Moe’s system that is needed to explain the true dynamics of the briefcase as it flies from the platform to Moe’s waiting hands.

## *The Coriolis Force*

How do the above ideas apply to the Earth? Because the Earth is rotating (i.e. accelerating) relative to the inertial frame of reference fixed to the stars, we can expect that accelerations arising out of this relative motion will appear as pseudo-forces in a coordinate system fixed to the earth. The Coriolis force is one such "pseudo-force."

To determine the *north-south* Coriolis effect on the ocean, consider the motion of a fluid parcel moving eastward with a speed u along a line of latitude, *f* , (Figure 5.18).

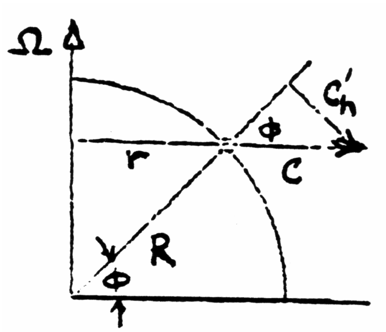



**Figure 5.18.** Components of the total centrifugal force on a water parcel.

The tangential speed of the Earth is

$$V = \Omega r = \Omega R \cos\phi$$

where Ω is the Earth rotation rate; R is the earth radius; and r is radius of gyration. The magnitude of the radial centrifugal force per unit volume on the water parcel is

$$c = r \left(\frac{V+u}{r}\right)^2$$

The horizontal component of c is

$$c_h = r \frac{(V+u)^2}{r} \sin\phi$$

$$= \underbrace{\frac{rV^{2}\sin f}{r}}_{\substack{\text{centrifugal}\\ \text{force at rest;}\\ \text{balanced by}\\ \text{earths ellipicity}}} + \underbrace{\frac{r\,2Vu\,\sin f}{r}}_{\substack{\text{Coriolis}\\ \text{" force"}}} + \underbrace{\frac{ru^{2}\sin f}{r}}_{\substack{\text{added centrifugal}\\ \text{force due to parcel}\\ \text{motion normally very}\\ \text{small compared to}\\ \text{Coriolis}}}$$

Thus the local, horizontal, northward component of the Coriolis force on the water parcel for this eastward moving parcel is

$$F_y^{\,Coriolis} = \frac{r\,2\Omega ru\,\sin f}{r} = r\,2\Omega \sin fu$$

$$= rfu \qquad \text{(southward)}$$

or

$$F_y^{\,Coriolis} = -rfu \qquad \text{(northward)}$$

where $f = 2\Omega \sin f$ .

The *east-west* Coriolis effect on the ocean can be shown through conservation of angular momentum arguments, that a water parcel moving northward at +v, will be “forced” eastward by a force per unit volume of

$$F_x^{\,Coriolis} = r\,2\Omega \sin f\, v = rfv \quad \text{(eastward)}$$

Thus, the total Coriolis force acting in the horizontal plane is

$$\vec{\mathrm{F}}^{\mathrm{Coriolis}} \approx r\ \mathrm{fv}\ \vec{\mathrm{i}} - r\ \mathrm{fu}\ \vec{\mathrm{j}}\,.$$

*****************************************************

The derivation of the *full* Coriolis force in Appendix B shows that

$$\sum \vec{\mathrm{F}}_{\mathrm{ext}} = \sum \vec{\mathrm{F}}_{\mathrm{rot}} + \mathrm{m}(2\vec{\Omega}\ \mathrm{x}\ \vec{\mathrm{v}}_{\mathrm{rot}}) + \mathrm{m}(\vec{\Omega}\ \mathrm{x}\ \vec{\Omega}\ \mathrm{x}\ \vec{\mathrm{r}})$$

or

$$\sum \vec{\mathrm{F}}_{ext} = \sum \vec{\mathrm{F}}_{rot} - \vec{\mathrm{F}}_{\mathrm{Coriolis}} - \vec{\mathrm{F}}_{\mathrm{centrifugal}}\ ,$$

where $\vec{\Omega}$ is the Earth rotation rate; $\sum \vec{F}_{ext}$ are the forces as seen by an observer in an inertial frame of reference; $\sum \vec{F}_{rot}$ and $\vec{v}_{rot}$ are the respective forces and velocity "seen" by an observer in the rotating frame of reference.

Thus in terms of the pseudo-forces - $\vec{F}_{Coriolis}$ and - $\vec{F}_{centrifugal}$ in the rotating frame of reference:

$$\sum \vec{F}_{rot} = \sum \vec{F}_{ext} + \vec{F}_{Coriolis} + \vec{F}_{centrifugal} .$$

The vector form of the Coriolis force is

$$\overrightarrow{F}_{Coriolis} = r\left(fv - f \cot f\, w\right)\vec{i} + r\left(-fu\right)\vec{j} + r\left(f \cot f\, u\right)\vec{k}$$
.

The full vector form of the Coriolis force can be simplified by comparing estimates of the different terms using "typical" oceanic quantities, namely

1) horizontal velocity difference; V
2) horizontal length scale; L 100 m < L < 1000 km
3) vertical velocity difference; W
4) vertical length scale; H; 1 km < H < 10 km

First, compare the relative sizes of u, v and w. This can be done by "scaling" the finite difference form of the continuity equation such that;

$$\frac{W}{H} + \frac{U}{L} = 0 \qquad ,$$

which implies that

$$W\, a\ \frac{H}{L} U .$$

that is the vertical velocity is typically diminished relative to U by the <u>aspect ratio</u> of the ocean H/L, which ranges $0.01 < H/L < 0.10$. Thus

$$W\ \alpha\left(0.10-0.01\right)U \quad ,$$

which means that W is small compared to U or W << U.

Therefore to a good approximation, the x-directed Coriolis component is

$$F_x^{Coriolis} = \rho\left(fv - f\cot\phi\, w\right) \;\approx\; \rho f v$$

.

How important is the vertical Coriolis component, $F_z^{Coriolis} = \rho f \cot\phi\ u$ compared to the gravitational component? First, away from the equator for typical latitudes, $\cot\phi$ is of order 1. So what is the size of fu? Estimating

$$2\Omega = 1.5 \text{ x } 10^{-4} \text{ rad/sec}$$

and

$$u = 10\text{-}100 \text{ cm/sec}$$

gives

$$\text{fu} = 1.5 \text{ x } 10^{-2} \text{ to } 1.5 \text{ x } 10^{-3} \frac{\text{cm}}{\text{sec}^2}$$

which is <u>very</u> small compared to g = 980 cm/sec$^2$.

Thus in most cases, $F_z^{Coriolis} = \rho f \cot\phi\ u \approx 0$ in comparison to $\rho$ g.

*Where do the above approximations break down?*

With these approximations, the relatively complex Coriolis formulation is reduced to our original estimate (pg 32)

$$\vec{F}^{\text{Coriolis}} \approx \rho \text{ fv } \vec{i} - \rho \text{ fu } \vec{j}\,.$$

with $f = 2\Omega\sin\phi$, which from Figure 5.19, is twice the local overhead rotation rate or $f = 2\Omega_v$.

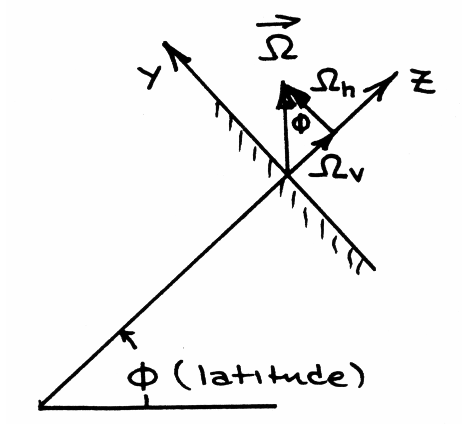



**Figure 5.19**. Earth rotation rate $\vec{\Omega}$ components in a local Cartesian coordinate system at latitude $f$., where $\Omega_v = \Omega \sin f = f/2$

Of course, that overhead rotation rate varies with latitude from + $\Omega$ at the north pole to - $\Omega$ at the south pole as illustrated in Figure 5.19

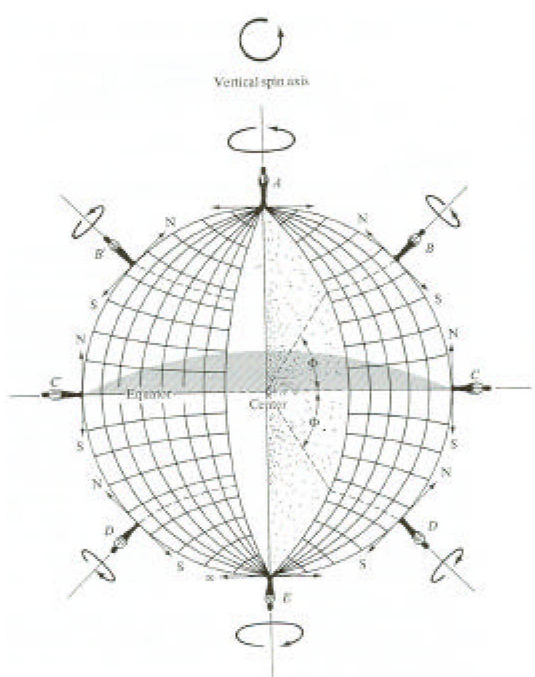



**Figure 5.20.** The latitudinal variation of the overhead Earth rotation rate $\Omega_v = \Omega \sin f$ as seen by a fixed observer.

******************************************************

With the above considerations we can write the following compact vector form of Newton's 2nd Law per unit volume for a fluid ocean

$$\rho \frac{d\vec{V}}{dt} = -\rho(2\vec{\Omega} \times \vec{V}) - \nabla p - \rho g\vec{k} + \vec{F}^f$$

as well as the following component form of this equation, in which we have neglected the less important Coriolis terms

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} + F_x^f$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} + F_y^f$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z^f$$

where $F_x^f = \left(\frac{\partial \tau_{zx}}{\partial z} + \left[\frac{\partial \tau_{yx}}{\partial y}\right]\right)$ ; $F_y^f = \left(\frac{\partial \tau_{zy}}{\partial z} + \left[\frac{\partial \tau_{xy}}{\partial x}\right]\right)$ $F_z^f = \left(\frac{\partial \tau_{yz}}{\partial y} + \left[\frac{\partial \tau_{xz}}{\partial x}\right]\right)$

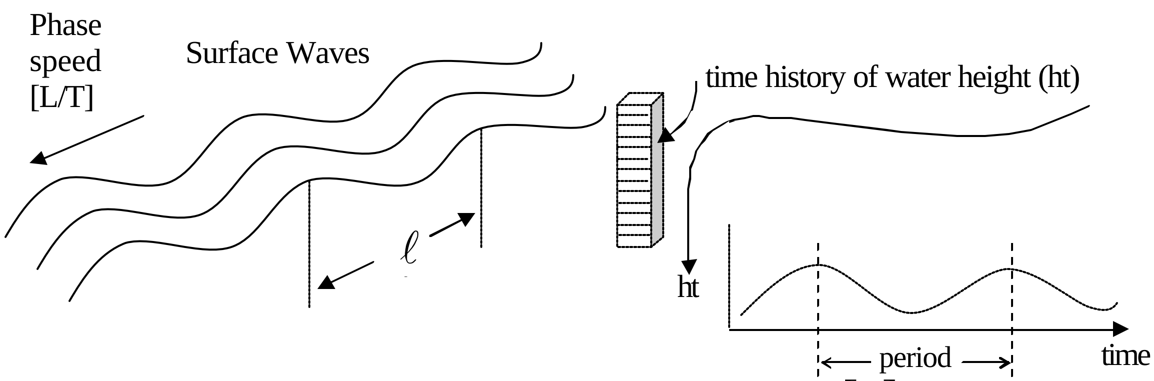
The latter will be very useful will be more useful for exploring dynamic balances of some of the more important oceanic processes.

## Chapter 5 - PROBLEMS

### Problem 5.1 Dimensional Analysis of Ocean Processes

a) Wave Speed: Given the following situation: deep, homogeneous (constant density) water of density $r\left[M/\mathrm{L}^3\right]$; force of gravity $\mathrm{g}\left[L/\mathrm{T}^2\right]$ ; and wave length $\ell[\mathrm{L}]$ where the [ ] indicate the generalized dimensions of time, T, length, L, and mass, M.
What must be the formula for the wave period ? ..............or phase speed?




b) Geostrophy: For many ocean flows only the velocity $\mathrm{u}\left[\frac{\mathrm{L}}{\mathrm{T}}\right]$, Coriolis coefficient $\mathrm{f}\left[\frac{1}{\mathrm{T}}\right]$, pressure difference $?\,\mathrm{p}\left[\mathrm{M}\frac{\mathrm{L}}{\mathrm{T}^2}\cdot\frac{1}{\mathrm{L}^2}\right]$ over a distance $\ell[\mathrm{L}]$ , and density $r\left[M/\mathrm{L}^3\right]$ are important for the dynamics.
How can the velocity of the flow be estimated from the other data?

c) Hydrostatic: Important pressure differences $\Delta p\left[M/L\,T^2\right]$ are related to the density anomaly $d\,r\left[M/\mathrm{L}^3\right]$ through gravity, $\mathrm{g}\left[L/\mathrm{T}^2\right]$ and the depth H[L].
How?

d) Find an expression for u, by eliminating $\Delta$p in the results from (b) and (c). Then given:

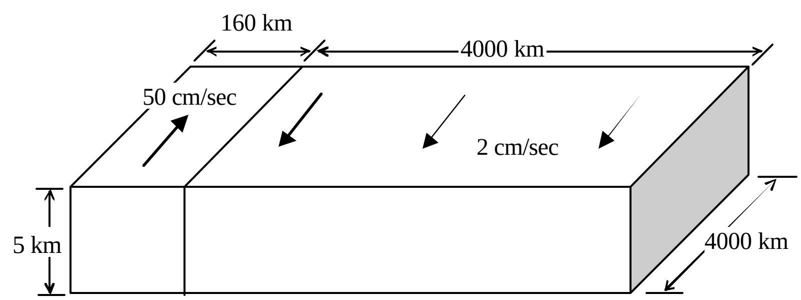
| | | |
|---|---|---|
| $d\,r$ | = | $10^{-3}$ gm/cm$^3$ |
| $r$ | = | 1 gm/cm$^3$ |
| f | = | $10^{-4}$ 1/sec |
| g | = | 980 cm/sec$^2$ |
| H | = | 3000 m |
| $\ell$ | = | 300 km |

what is v for these circumstances?

**Problem 5.2 Kinetic Energy**

Consider the following estimate of oceanic kinetic energy. The model ocean is 5 km deep and currents extend uniformly to the bottom. All water within each section is moving at the indicated velocity.




a) What is the total kinetic energy KE of the system expressed in ergs and joules?

<u>Hint</u>: The system kinetic energy is KE = $\sum_i 1/2\, M_i V_i^2$, where $M_i \equiv$ mass of the $i^{th}$ particle; $V^i \equiv$ velocity of the $i^{th}$ particle; and $\sum_i$ means summation over all particles.

b) How much heat energy (in calories) would be released if all the system kinetic energy were converted to heat?

How many minutes of sunlight at 0.25 ly/min would supply this much heat to the ocean surface?

### Problem 5.3 Estimation of Vertical Velocity

The mean horizontal current components (cm/sec) in the upper 50 m of an ocean have been averaged over five-degree squares and are shown in the figure below.

a) Estimate the vertical velocity at 50 m in each of the five-degree squares.

b) What assumption have you made, and why may your answer be erroneous?

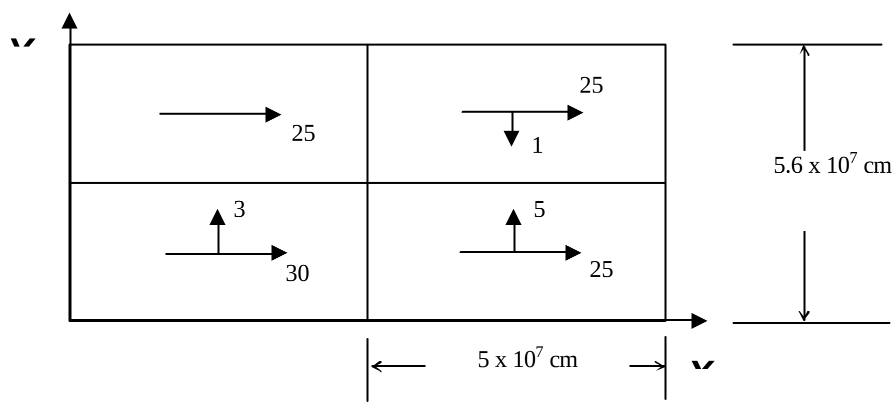
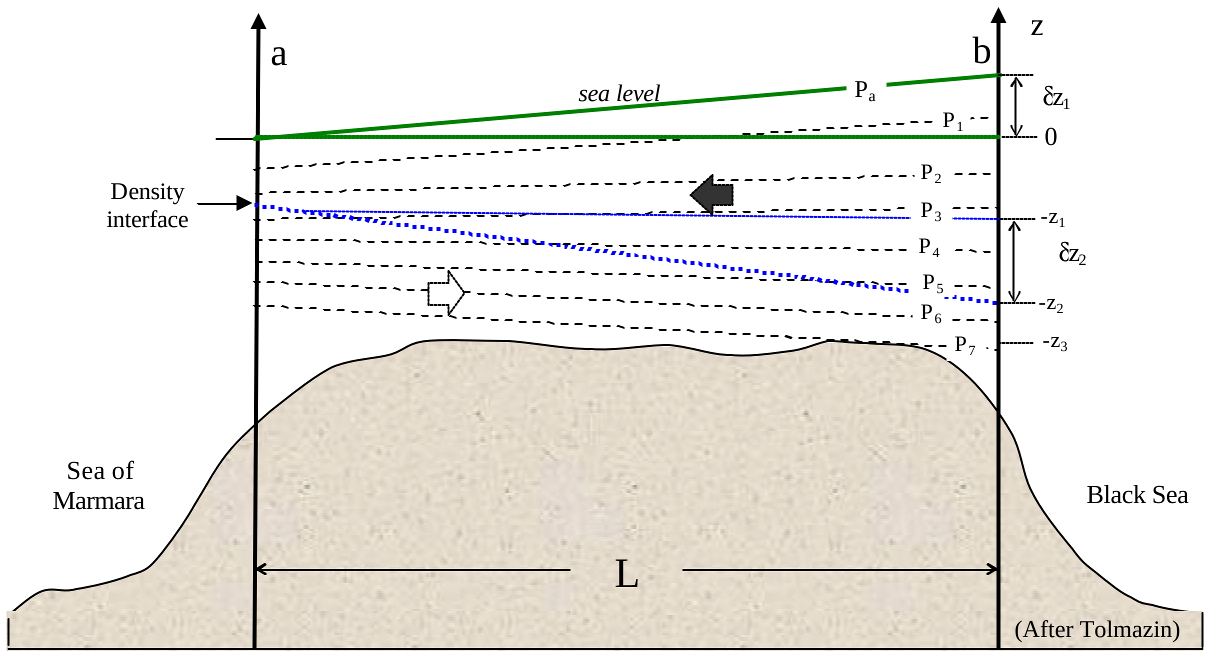



**Figure:** Horizontal current components, averaged for four 5-degree squares.

**Problem 5.4 Horizontal Pressure Gradient Force**

Below is a schematic representation of the geometry and pressure field in a longitudinal section of the Strait of Bosporus. The ocean surface elevation at $d\ z_1$, above the mean sea level (at z = 0) coincides with the atmospheric pressure ($P_a$) isobar. The slopes of this and the isobars at depth (i.e. $P_1$, $P_2$, $P_3$, etc.) are grossly exaggerated compared to the slope of the bold, dashed interface between the low density ( $r_B$) Black Sea and the high density ( $r_m$) Sea of Marmara waters.




Nevertheless these slopes are real because of the density differences in Sea of Marmara ($1.020 < r_m < 1.025$ gm/cm$^3$) and the Black Sea ($1.012 < r_B < 1.016$ gm/cm$^3$). Show that the pressure gradient forces at the different depths are consistent with the flow directions as pictured.

a) Derive the relation for the horizontal pressure gradient force (PGF) under these circumstances from the definition:

$$PGF = -dp/dx = -(P_b - P_a)/L \qquad ,$$

where the subscripts a and b refer to the two sections in the picture above.

b) Evaluate the PGF at depths z = 0, $-z_1$, $-z_2$, and $-z_3$ respectively.

c) What are the slopes of isobars $P_a$, $P_3$, and $P_6$ respectively, assuming $r_m = 1.025$ gm/cm$^3$ and $r_B = 1.012$ gm/cm$^3$.

## Problem 5.5 Pseudo Forces

a) Describe as briefly as possible the distinction between centrifugal and Coriolis forces.

b) Why are these forces called pseudo forces?

c) What is suggested by the fact that these pseudo forces and the gravitational forces are both proportional to mass?

d) What is the centrifugal force exerted on a typical 50 kg person at latitude 43°N? What is the value of the Coriolis parameter f in units of $sec^{-1}$? If the person runs the 30 m dash in 10 sec, what is the Coriolis force felt by the person? What is the respective ratios of these two “forces” to the person's weight, mg?

e) An aircraft flying eastward at u = 1000 nautical miles/hour at latitude 60°N relative to a rotating earth. Neglecting second order effects (e.g. $u^2$), compute the deflection of local apparent vertical as observed by a person on the airplane. (Recall the analysis of Chapter 1 in the notes). How much does the inclusion of the second order term change this result?