Rate Distortion Bounds on Passive Sonar Performance

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***Abstract*— Information theory provides a novel perspective on passive sonar performance analysis. This approach begins by partitioning the search space and then considers the problem of assigning an unknown source to the correct partition based on pressure observations from a hydrophone array. Prior work described necessary conditions for achieving arbitrarily small probability of error (P_e) as a tradeoff between SNR and the range precision of the partitions. The current work presents a method to extend these results using the rate distortion function to find necessary conditions for any P_e, not just arbitrarily small ones. The Gaussian channel bound sets an upper limit on the information rate received at the array. Through the rate distortion function, this upper limit on information rate implies a lower bound on P_e for a given partition. Furthermore, the current work describes a tradeoff among range precision, P_e and SNR. Examples of this tradeoff are given for a typical 2-dimensional (range and depth) shallow water environment.**

I. Introduction

Passive sonar estimates the location of a sound source from pressure observations at a hydrophone array and a model of acoustic propagation in the environment. Traditionally, passive sonar performance is quantified by the *Mean Squared Error* (MSE) between the true source location and its estimate. Examples of bounds on the MSE performance in passive sonar include the *Cramer Rao Bound* (CRB) for matched field processing [1], the *Extended Ziv-Zakai Bound* (EZZB) for bearing estimation [2], and Bayesian bounds for matched field parameter estimation [3].

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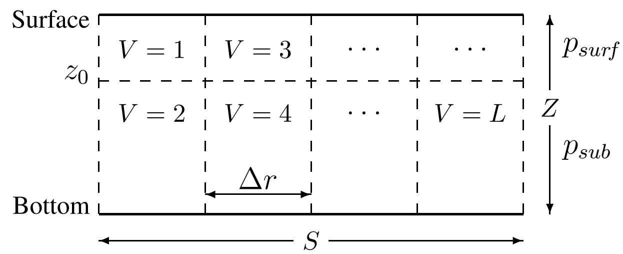

Fig. 1. Search space.

In 2002, Buck proposed the information theoretic bound [4]. This approach begins by dividing the search space into partitions and then considers the problem of assigning an unknown source to the correct partition with the minimum possible *probability of error* (P_e) based on the pressure observations from a hydrophone array. Fig. 1 shows an example search space with range S and depth Z. Sources above z_0 are considered as surface ships with probability p_{surf} and sources below z_0 are considered as submarines with probability $p_{sub} = 1 - p_{surf}$. All ranges are equiprobable. Each partition has the same width in range, Δr, called the range precision.

Ref. [4] treated the narrowband passive sonar problem as an unorthodox communication problem, as shown in Fig. 2. A source with location W is contained in partition V. $X(W)$ is the baseband sound pressure observed at the hydrophone array for a source at location W. $Y(X)$ is the noisy observation, due to the additive Gaussian noise N, obtained by the hydrophone array. Any passive sonar system uses the observations Y to estimate

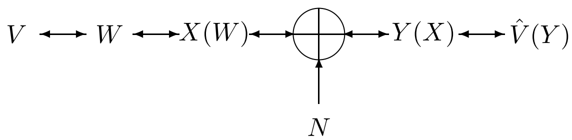

Fig. 2. Communication channel.

the partition V containing the source, attempting to minimize $P_e = \Pr\{V \neq \hat{V}\}$, where $\hat{V}(Y)$ is the estimated partition.

According to the source channel coding theorem [5], to achieve arbitrarily small P_e, the mutual information between V and $\hat{V}$ must be greater than or equal to the entropy of V, i.e., $P_e < \epsilon \Longrightarrow I(V;\hat{V}) \geq H(V)$.

The Gaussian channel upper bound [5] puts an upper limit, I_{max}, on the mutual information between the input, X, and the output, Y, of an additive Gaussian channel. This upper limit, I_{max}, is

$$I_{max} \;=\; \log_2\left(\frac{|\mathbf{K}_{xx}+\mathbf{K}_{nn}|}{|\mathbf{K}_{nn}|}\right), \qquad (1)$$

where $\mathbf{K}_{xx}$ and $\mathbf{K}_{nn}$ are the covariance matrices of the sound wave pressure, X, and the noise, N, respectively. Because the vectors X and Y are complex, this equation does not have the leading $1/2$ commonly seen in the bound for real channels.

The data processing theorem [5] indicates that $I(V;\hat{V}) \leq I(X;Y)$ and thus connects together the results from the source channel coding theorem and the Gaussian channel upper bound: $H(V) \leq I(V;\hat{V}) \leq I(X;Y) \leq I_{max}$, or $H(V) \leq I_{max}$.

For arbitrarily small P_e, $H(V) \leq I_{max}$ implies a necessary condition on the range precision Δr:

$$\Delta r \;\geq\; S \cdot 2^{H_D} \cdot \left(\frac{|\mathbf{K}_{nn}|}{|\mathbf{K}_{xx}+\mathbf{K}_{nn}|}\right), \qquad (2)$$

where $H_D = H(p_{surf}, p_{sub})$, the entropy of the binary probability distribution $\{p_{surf}, p_{sub}\}$.

The example in [4] evaluates Eq. (2) in a shallow water environment to plot the range precision bound versus the *Signal-to-Noise Ratio* (SNR), or the SNR bound versus the desired range precision, depending on how one interprets the result.

A natural question raised by the information theoretic bound is how fast P_e increases if the desired range precision, or SNR, goes below the lower bound? Another question is how the range precision lower bound changes if larger P_e's are allowed at a given SNR? These questions motivated the current work.

II. Approach

The rate distortion function $R(D)$ is required to accommodate P_e's that are not arbitrarily small. For a given probability distribution $p(v)$,

$$R(D) = \min_{p(\hat{v}|v):\sum_{(v,\hat{v})} p(\hat{v}|v)p(v)d(v,\hat{v}) \leq D} I(V;\hat{V}), \quad (3)$$

where $p(\hat{v}|v)$ is the conditional probability distribution of $\hat{V}$ given V, $d(v,\hat{v})$ is the distortion function, and D is a constraint on the expected distortion [5].

The current work uses the Hamming distortion function:

$$d(v,\hat{v}) \;=\; \begin{cases} 0, & v = \hat{v}, \\ 1, & v \neq \hat{v}. \end{cases} \qquad (4)$$

By this definition, $\sum_{(v,\hat{v})} p(\hat{v}|v)p(v)d(v,\hat{v}) = \Pr\{V \neq \hat{V}\} = P_e$. Therefore D is a constraint on P_e, and $R(D)$ is the minimum mutual information between V and $\hat{V}$ required to achieve $P_e \leq D$. Although in general there is no simple expression for $R(D)$, it can be calculated analytically [6]. On the $R(D)$ curve, $P_e = D$ [6], so we will use $R(P_e)$ for the rate distortion function below.

The derivation of the rate distortion bound is very similar to that of the information theoretic bound. Replacing $H(V)$ with $R(P_e)$ results in the inequality $R(P_e) \leq I_{max}$, which is an extended version of $H(V) \leq I_{max}$ since $R(P_e) = H(V)$ for arbitrarily small P_e. Fig. 3 shows a sample $R(P_e)$ curve. When $P_e = 0$, $R(P_e) = H(V)$. When $P_e > 1 - p_{max}$, where $p_{max} = \max p(v)$ is the probability of the most probable partition, the information rate provided by the channel could be as low as zero since one can always choose the most probable partition and P_e won't be greater than $1 - p_{max}$. The $R(P_e)$ curve decreases monotonically as P_e increases [6], which is intuitively reasonable since as more errors are allowed less information is required. Given I_{max} from the Gaussian channel bound (indicated by the horizontal dashed line on Fig. 3), the information rate R must be below that bound. According to the $R(P_e)$ curve, P_e must then be greater than the value indicated by the vertical

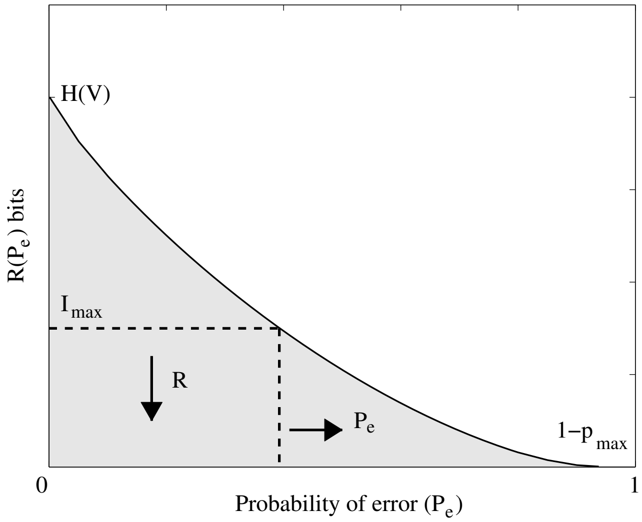

Fig. 3. $R(P_e)$ curve.

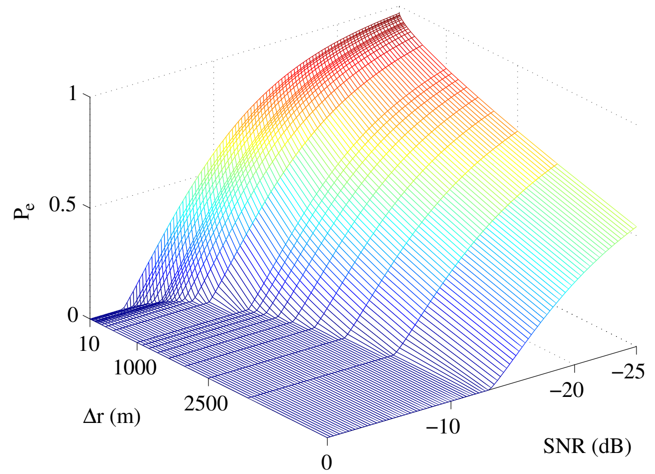

Fig. 5. Tradeoff surface. The volume under the surface is unachievable.

dashed line. So $R(P_e) \leq I_{max}$ implies a necessary condition on P_e versus I_{max}: the shaded region under the $R(P_e)$ curve is unachievable, i.e., for any point in that region the corresponding I_{max} and P_e cannot satisfy $R(P_e) \leq I_{max}$. Because this condition is necessary but not sufficient, the region above the $R(P_e)$ curve is not guaranteed to be achievable.

The $R(P_e) \leq I_{max}$ condition may be related to the passive sonar problem to describe a tradeoff among the range precision, P_e and SNR, as shown in Fig. 4. If the underwater acoustic environment, the hydrophone array geometry, the probabilities of surface ships and submarines, and the noise are all fixed, I_{max} is a function of SNR and the information rate, R, is a function of range precision, Δr, and P_e. Through $R(P_e) \leq I_{max}$, the range precision, P_e and SNR form a tradeoff surface. Fig. 5 shows an example of the tradeoff surface, the volume under which is unachievable, i.e., for any point under the tradeoff surface, the corresponding range precision, P_e and SNR cannot satisfy $R(P_e) \leq I_{max}$.

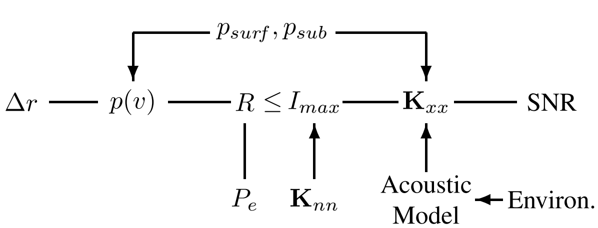

Fig. 4. Tradeoff among range precision, P_e and SNR.

Cutting the surface in different ways, one can obtain different lower bound curves: (i) For fixed SNR, one can have a set of P_e versus range precision curves corresponding to different SNR values, which show how fast P_e increases when the desired range precision becomes less than its lower bound; (ii) For fixed range precision, one can have a set of P_e versus SNR curves corresponding to different range precision values, which show how fast P_e increases when SNR goes below its lower bound; (iii) For fixed P_e, one can have a set of range precision versus SNR curves corresponding to different P_e values, which show how the range precision bound changes with P_e. Therefore, the tradeoff among range precision, P_e and SNR described by these rate distortion bounds addresses the questions motivating the current work.

III. Simulation Results

This section applies the results from Section II to an example passive sonar problem. The 2–dimensional (range and depth) shallow water acoustic environment and vertical hydrophone array geometry used in this example are shown in Fig. 6, the same environment and array as in the example in [4]. The acoustic model used to simulate the underwater acoustic environment is OASES [7]. The noise is spatially white. The ratio of surface ships to submarines is $10:1$, i.e., $p_{surf} = 10 p_{sub}$. Figs. 7–9 plot the resulting rate distortion bounds.

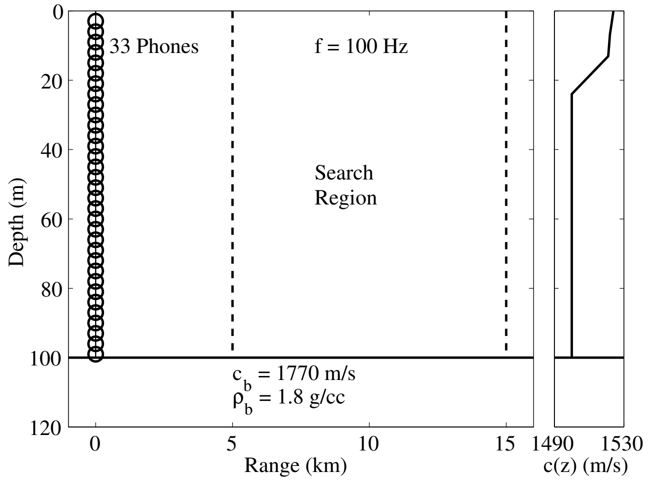

Fig. 6. Acoustic environment and hydrophone array geometry. A vertical hydrophone array with 33 phones equally spaced from 3 m to 99 m in depth is located at 0 km in range. A source with known amplitude but unknown phase is in the search region which is from 0 to 100 m in depth and from 5 km to 15 km in range. The source is radiating sound energy at a single frequency 100 Hz. The bottom is an infinite fluid half-space with sound speed $c_b = 1770$ m/s and density $\rho_b = 1.8$ g/cc. The sound speed profile for the shallow water environment is shown at the right part of the above figure.

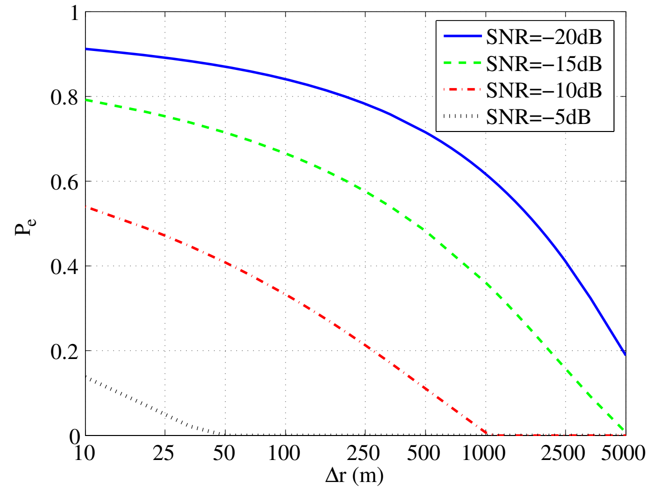

Fig. 7. P_e versus range precision bounds for SNR $= -20$ dB (solid line), -15 dB (dashed line), -10 dB (dashdot line) and -5 dB (dotted line). The region under each curve is unachievable.

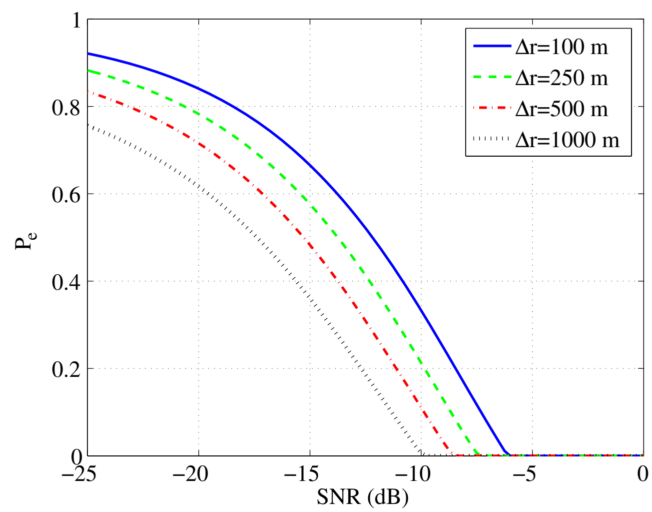

Fig. 8. P_e versus SNR bounds for given range precisions: $\Delta r =$ 100 m (solid line), 250 m (dashed line), 500 m (dashdot line), and 1000 m (dotted line). The region under each curve is unachievable.

Fig. 7 shows P_e versus range precision bounds for several given SNRs. The region under each curve is unachievable. As SNR increases, the area of the unachievable region becomes smaller and smaller toward the bottom–left corner, which implies that (i) for a desired range precision, a lower P_e may be achieved as SNR increases; or, (ii) for a desired P_e, a higher range precision, or smaller Δr, may be achieved as SNR increases. For example, if the desired range precision is 250 m, P_e cannot be less than about 0.8 for SNR$= -20$ dB, about 0.6 for SNR$= -15$ dB, about 0.2 for SNR$= -10$ dB, while P_e could be arbitrarily small for SNR$= -5$ dB; if the desired P_e is 0.2, the range precision cannot be better than about 5000 m for SNR$= -20$ dB, about 2000 m for SNR$= -15$ dB, about 250 m for SNR$= -10$ dB, while the range precision could be better than 10 m for SNR$= -5$ dB. Note that the intersection of each curve with the $P_e = 0$ line is actually the information theoretic bound on range precision from [4] for the corresponding SNR.

Fig. 8 shows P_e versus SNR bounds for several given range precisions. The region under each curve is unachievable. As the range precision becomes lower, i.e., Δr increases, the area of the unachievable region becomes smaller and smaller toward the bottom–left corner, which implies that (i) for a given SNR, a lower P_e may be achieved as Δr increases; or, (ii) for a given P_e, a larger Δr requires less SNR. Note that the intersection of each curve with the $P_e = 0$ line is again the information theoretic bound on SNR from [4] for the corresponding range precision.

Fig. 9 shows range precision versus SNR bounds for several given P_e's. The region under each curve is unachievable. As the allowed P_e increases, the

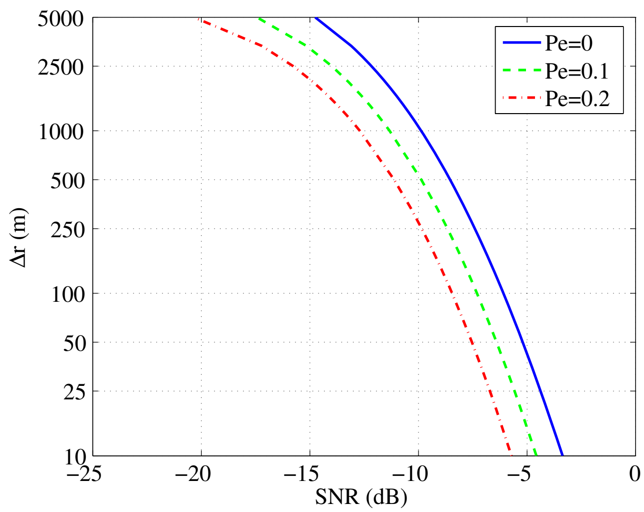

Fig. 9. Range precision versus SNR bounds for $P_e = 0$ (solid line), 0.1 (dashed line), and 0.2 (dashdot line). The region under each curve is unachievable.

area of the unachievable region becomes smaller and smaller toward the bottom–left corner, which implies that (i) for a given SNR, better range precision may be achieved as more error is allowed; or, (ii) for a desired range precision, less SNR is required as a larger P_e is allowed. Note that the solid line corresponding to $P_e = 0$ is once again the information theoretic bound from [4].

IV. Summary and Discussion

The rate distortion bound generalizes the information theoretic bound from [4] using the rate distortion function in place of the partition entropy $H(V)$. The rate distortion bound quantifies the tradeoff among range precision, P_e and SNR through the tradeoff surface. Like the Ziv–Zakai and Weiss–Weinstein bounds in [2] and [3], the rate distortion bound is a Bayesian bound because it requires the knowledge of the source probability distribution.

Compared to the Ziv–Zakai and Weiss–Weinstein bounds, the rate distortion bound is relatively easy to evaluate. $R(P_e)$ can be calculated analytically for a given source distribution. To obtain I_{max}, one only needs to know the signal and noise covariance matrices $\mathbf{K}_{xx}$ and $\mathbf{K}_{nn}$. In contrast, sophisticated and time–consuming numerical integrations are required to evaluate the Ziv–Zakai and Weiss–Weinstein bounds for realistic passive sonar scenarios.

Many passive sonar scenarios have logistic considerations motivating the rectangular partition used in this work. This partitioning scheme almost certainly is not the best geometry for minimizing P_e for a given partition PDF $p(v)$. It is an important open question to find which partition with a given $p(v)$ minimizes P_e, and how tightly the limits presented here bound the performance of that partition.

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