Fading Channel Capacity and Passive Sonar Performance Bounds

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***Abstract*— Passive sonar algorithms attempt to estimate the location of a sound source from pressure observations at a hydrophone array and knowledge of the acoustic environment. Traditionally, the mean squared error (MSE) of these position estimates has been the performance metric of interest, and performance bounds set lower limits on the MSE. Information theory provides an alternative perspective on passive sonar performance. In this approach, the search volume is partitioned into disjoint cells, each of which has an *a priori* probability of containing the source. The sonar algorithm then estimates which partition cell contains the unknown source based on the array observations. The goal is to minimize the probability of error (P_e) in making this assignment. A necessary condition to achieve arbitrarily small P_e is that the mutual information between the actual and estimated cell containing the source must exceed the entropy of the cells computed from the *a priori* probabilities. For a fixed size search region, this necessary condition implies a lower bound on the cell size which can achieve arbitrarily small P_e. If the average source level is known, but not the individual source levels, the mutual information is bounded by the capacity of a fading channel. The cell sizes satisfying the necessary condition for arbitrarily small P_e for the fading channel model are substantially larger than those for the case when the source level is known. The resulting performance bound is evaluated for a typical coastal shallow water environment.**

I. Introduction

Passive sonar performance is commonly quantified in terms of the mean squared error (MSE) between the estimated and true source position. Performance limits are then lower bounds on the achievable MSE, such as the Cramer-Rao Lower Bound (CRLB) [1], the Ziv-Zakai Bound (ZZB) [2]–[4], and the Weiss-Weinstein Bound (WWB) [4], [5]. The CRLB is based on the curvature of the local likelihood peak, and thus often suffers in passive sonar problems where large errors due to ambiguities often dominate. The ZZB and WWB are Bayesian bounds incorporating the global structure of the likelihood surface, but are computationally demanding to evaluate in realistic underwater acoustic environments [4].

An information theory perspective on passive sonar performance limits reframes the error metric as the probability of error (P_e), instead of the MSE. In many passive sonar scenarios, logistical considerations suggest partitioning the search volume as shown in Fig. 1 for a two-dimensional version of the problem. All contacts above some threshold depth z_0 are considered surface contacts; those below z_0 are considered submerged contacts. The range is divided into bins of width Δr. This range bin width is referred to as the range accuracy below. Although this paper does not consider bearing, the bound presented is straightforward to extend to that dimension as well. Previous work [6] derived necessary conditions on achieving arbitrarily small P_e (ASP) with a given array geometry in an environment using the source-channel coding theorem and the Gaussian channel upper bound [7]. These conditions yielded a tradeoff between the minimum Δr and the signal-to-noise ratio (SNR) in an environment. A shortcoming of this bound was the implicit assumptions that all contacts had the same source level and that the source level was known exactly. A more realistic assumption is that only the average source level is known for an ensemble of contacts, and that the source level of each contact is independent of all the others. Under this new assumption, bounding the

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passive sonar performance for a source of unknown level in a known channel is isomorphic to finding the capacity of a fading communications channel with a known source level but random channel gain. The resulting bound is Bayesian in structure, but significantly easier to evaluate for a given environment than the WWB and ZZB bounds.

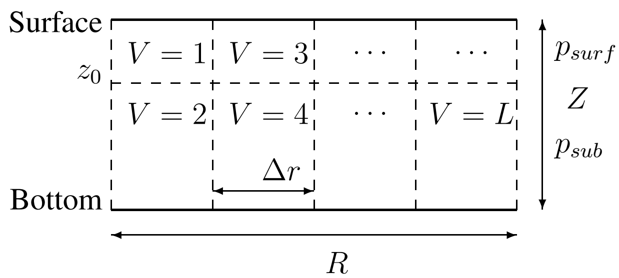

Fig. 1. Rectangular partition for passive sonar search space

II. Bound Formulation

To establish notation, let $\mathbf{W} = (r, z)$ be the range and depth of an unknown single-frequency source, with probability density function (PDF) $p_{\mathbf{W}}(\mathbf{w})$. Each source position is assigned to a partition $V = 1, \ldots, L$ according to $V(\mathbf{W})$, inducing a partition PDF $p_V(v)$. Let $\mathbf{X}(\mathbf{W})$ be the vector of basebanded phasors representing the single frequency pressure field observed at the hydrophone array for a source at position $\mathbf{W}$, and $\mathbf{Y} = \mathbf{X}(\mathbf{W}) + \mathbf{N}$ be the noisy observations, where $\mathbf{N}$ represents zero-mean complex Gaussian noise with covariance $\mathbf{K_{NN}}$ which is independent of $\mathbf{X}$. The estimated partition is $\hat{V}(\mathbf{Y})$, and the performance metric $P_e = \Pr\{\hat{V} \neq V\}$. As developed in [6], the goal is to formulate a lower bound on Δr satisfying the necessary conditions for ASP.

A necessary condition to achieve ASP is that the entropy of the partition must be less than or equal to the mutual information between the partition V and its estimate $\hat{V}$, i.e., $H(V) \leq I(V, \hat{V})$. Because both $\mathbf{X}$ and V are deterministic functions of $\mathbf{W}$, and $\hat{V}$ is a deterministic function of $\mathbf{Y}$, the data processing theorem (Thm. 2.8.1 of [7]) can be used to show that $I(\mathbf{X}, \mathbf{Y}) \geq I(V, \hat{V})$. Ref. [6] then applied Gaussian channel upper bound to find an upper bound on $I(\mathbf{X}, \mathbf{Y})$ in terms of the replica covariance $\mathbf{K_{XX}} = E_{\mathbf{W}}\{\mathbf{X}\mathbf{X}^H\}$ and the noise covariance $\mathbf{K_{NN}}$. For a complex additive Gaussian channel, the Gaussian channel upper bound on the mutual information between $\mathbf{X}$ and $\mathbf{Y}$ is

$$\begin{aligned} I(\mathbf{X}, \mathbf{Y}) &\leq \log_2(|\mathbf{K_{YY}}|/|\mathbf{K_{NN}}|), \\ &= \log_2(|\mathbf{K_{XX}} + \mathbf{K_{NN}}|/|\mathbf{K_{NN}}|), \end{aligned} \quad (1)$$

where the second line exploits the independence of $\mathbf{X}$ and $\mathbf{N}$. Eq. (1) lacks the leading factor of $1/2$ commonly seen in this bound for real additive Gaussian channels because in our model the channel is complex with equal noise in both quadrature components. The two quadrature components provide twice the mutual information of a real channel with the same total SNR. Note that when all of the factors in the problem other than the source level are fixed, the bound in Eq. (1) can be parameterized in terms of the overall SNR, where

$$\text{SNR} = \text{trace}\{\mathbf{K_{XX}}\}/\text{trace}\{\mathbf{K_{NN}}\}.$$

Linking the inequalities yields that

$$\log_2(|\mathbf{K_{XX}} + \mathbf{K_{NN}}|/|\mathbf{K_{NN}}|) \geq I(\mathbf{X}, \mathbf{Y}) \geq H(V) \quad (2)$$

is a necessary condition for ASP, and this relation can be solved to provide a lower bound on Δr. The partition entropy associated with the depth dimension is $H_z = H(p_{surf}, p_{sub})$, the binary entropy for the relative probability of each kind of contact. If all ranges are equally likely, and range r and depth z are independent, then

$$H(V) = \log_2(R/\Delta r) + H_z. \quad (3)$$

Substituting Eq. (3) into Eq. (2) allows us to solve for the lower bound

$$\Delta r \geq R\, 2^{H_z}(|\mathbf{K_{NN}}|/|\mathbf{K_{XX}} + \mathbf{K_{NN}}|). \quad (4)$$

Any partition with Δr violating this bound will have P_e bounded away from 0. Because the bound is based on necessary but not sufficient conditions, there is no guarantee that any given partition satisfying the bound will achieve ASP.

A shortcoming of the development above is that it presumes knowledge of the source level in bounding $I(\mathbf{X}, \mathbf{Y})$. The model in Fig. 2 represents a less restrictive assumption. In this model, the source level b is a Gaussian random variable with known variance σ_b^2. Given the source variance σ_b^2 and the replica covariance structure $\mathbf{K_{XX}}$, the challenge is to find an upper bound on $I(\mathbf{X}, \mathbf{Y})$ that is tighter than Eq. (1). Such a bound will replace the Gaussian channel upper bound in Eq. (2), giving a tighter lower bound on the achievable Δr.

Abou-Faycal *et al.* [8] demonstrated that the real scalar memoryless Rayleigh-fading channel has capacity equivalent to an exponential channel. More importantly, they proved that the optimal source PDF for such a channel consists of a finite number of discrete mass points. Under this condition on the source PDF, the mutual information $I(\mathbf{X}, \mathbf{Y})$ for any set of mass points and associated probabilities may be numerically evaluated using Gauss-Laguerre integration [9]. Given a fixed number of mass points and an SNR constraint a/σ_n^2, a numerical gradient ascent search can then be used to find the optimal mass point locations and probabilities maximizing $I(\mathbf{X}, \mathbf{Y})$ while satisfying the power constraint $E\{X^2\} \leq a$. This fading channel upper bound on $I(\mathbf{X}, \mathbf{Y})$ is

$$I_{FC}(a) = \sup_{E\{X^2\} \leq a} I(\mathbf{X}, \mathbf{Y}) \tag{5}$$

Abou-Faycal *et al.* found heuristically that maximizing Eq. (5) for the source distribution with two mass points was within 5% of the maximum over any number of mass points for SNRs of 10 dB or less. As shown in Figure 3, the resulting capacity for the fading channel is significantly lower than the additive Gaussian channel at moderate SNRs and above. As shown below, this reduction of information available implies a larger lower bound on Δr.

Abou-Faycal *et al.*'s scalar results can be extended to bound $I(\mathbf{X}, \mathbf{Y})$ for the parallel channel model representing passive sonar arrays. For the spatially white (SW) noise case with $\mathbf{K_{NN}} = \sigma_n^2 \mathbf{I}$, the eigenvectors of $\mathbf{K_{XX}}$ form a similarity transform which decorrelates the vector problem into M uncorrelated parallel complex channels, where M is the number of nontrivial eigenvalues in $\mathbf{K_{XX}}$. In shallow water acoustic environments, M is the number of trapped acoustic modes propagating into the far field. Each of the $i = 1, \ldots, M$ uncorrelated channels has an SNR of $a_i = \lambda_i(\mathbf{K_{XX}})/\sigma_n^2$, where $\lambda_i(\mathbf{K_{XX}})$ is the ith eigenvalue of the signal covariance $\mathbf{K_{XX}}$. Exploiting Eq. (5) and accounting for

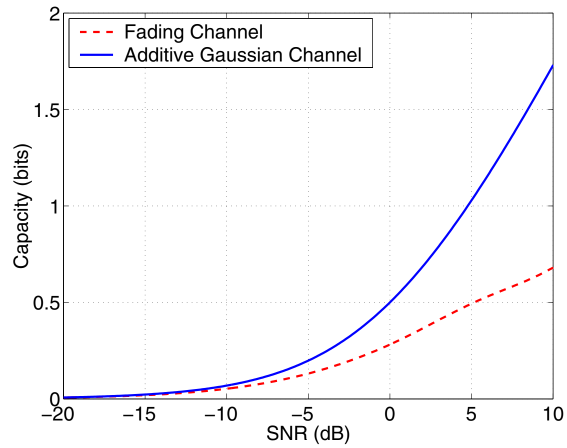

Fig. 3. Comparison of channel capacities for a real additive white Gaussian noise channel (solid) and a real memoryless Rayleigh-fading channel (dashed).

both quadrature components, the information in the ith channel is bounded above by $2I_{FC}(a_i)$, and the total mutual information by

$$I(\mathbf{X}, \mathbf{Y}) \leq \sum_{i=1}^{M} 2I_{FC}(a_i). \tag{6}$$

Replacing the determinants in Eq. (2) with the upper bound in Eq. (6) yields the fading channel bound on passive sonar range accuracy for SW noise environments:

$$\Delta r \geq R\, 2^{H_z} \left(\frac{1}{4}\right)^{\sum_{i=1}^{M} I_{FC}(a_i)}, \tag{7}$$

where H_z is the depth entropy as in Eq. (3). As expected, increasing the mutual information $2\sum I_{FC}(a_i)$ reduces the lower bound on Δr.

Extending the results in Ref. [8] to the Kuperman-Ingenito (KI) noise model [10] follows a similar approach to the SW noise model analysis in the previous paragraph. The one difference is that the similarity transform uses the eigenvectors of the noise covariance $\mathbf{K_{NN}}$ to decorrelate the vector problem into M parallel complex channels. The eigenvectors of $\mathbf{K_{NN}}$ will also diagonalize, or nearly diagonalize modulo some small spatial sampling effects, $\mathbf{K_{XX}}$, since Ref. [10] models the surface generated noise as propagating independently in each trapped mode. Once the similarity transform is applied to both $\mathbf{K_{XX}}$ and $\mathbf{K_{NN}}$, the ith channel

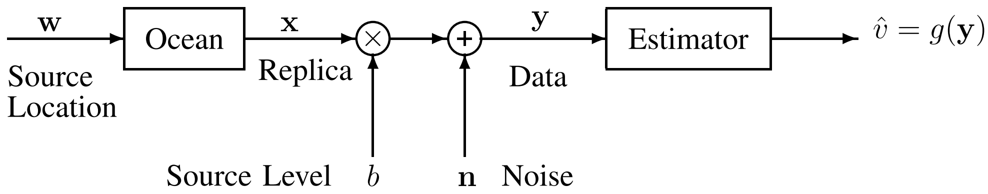

Fig. 2. Fading channel model for passive sonar

has a SNR of

$$a_i = \left[\mathbf{A}_n^H \mathbf{K}_{\mathbf{XX}} \mathbf{A}_n\right]_{ii} / \lambda_i(\mathbf{K}_{\mathbf{NN}}), \tag{8}$$

where $\mathbf{A}_n$ is the unitary matrix of eigenvalues of $\mathbf{K}_{\mathbf{NN}}$, $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operator, and $\lambda_i(\mathbf{K}_{\mathbf{NN}})$ is the ith eigenvalue of the noise covariance $\mathbf{K}_{\mathbf{NN}}$. Using the new definition of a_i in Eq. (8), Eq. (7) also describes the lower bound on the range accuracy Δr for the KI noise model.

III. Sample Bound Evaluation

Fig. 4 illustrates a typical shallow water environment. The source radiates $f = 100$ Hz and is equally likely to be a surface or submerged contact, i.e., $p_{surf} = p_{sub} = 1/2$. All ranges from 5–15 km are equally likely. A vertical receiving array of 33 hydrophones spaced every 3 m spans the water column. The downward-refracting sound speed profile shown in the right panel is assumed for all ranges within the environment. The replica covariance matrix $\mathbf{K}_{\mathbf{XX}}$ is computed for this acoustic environment using the OASES underwater acoustic simulation package [11] to evaluate the replica vectors every 10 m in range and 2 m depth. The performance bound in Eq. (7) is evaluated for the two common noise models discussed above.

Fig. 5 illustrates the passive sonar performance bounds obtained by evaluating Eqs. (4) and (7) for this environment with the SW noise model as the hydrophone level SNR varies from -20 dB to 0 dB. The values for I_{FC} in Eq. (7) were computed for the two mass point approximation every 0.1 dB from -20 dB to 10 dB, and then interpolated between these values as needed. Fig. 5 plots the lower bounds on Δr vs. SNR to achieve ASP for the unknown source level (solid line), and the known source level (dashed line) cases. Any combination of Δr and SNR below this line violates the necessary conditions for ASP, and is thus unachievable. For

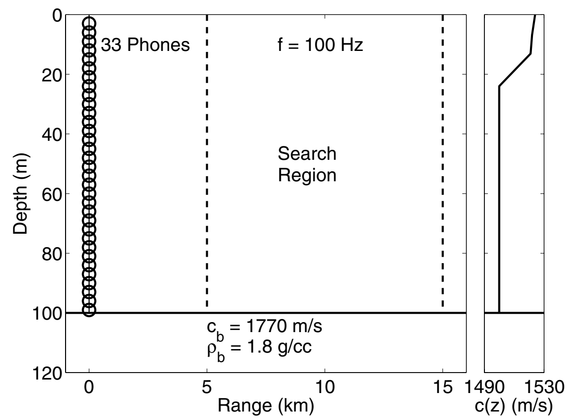

Fig. 4. Typical coastal environment with downward refracting soundspeed profile and vertical hydrophone array.

a nominal range accuracy of $\Delta r = 250$ m, the required SNR increases by roughly 7 dB when the source level is unknown, from roughly -7 dB for the known source level case in [6] to just under 0 dB for the fading channel bound from Eq. (7).

The range accuracy bound was also evaluated for the KI noise model using Eqs. (7) and (8) for SNRs in the range -20 dB to -8 dB. Mode 1 in the KI noise model has substantially higher values of a_i in Eq. (8) than for the SW noise model at the same SNR. Limiting the SNR to -8 dB insures that the maximum value of a_i in the KI model is about 10 dB, which was the upper limit on a_i for the SW model for SNR = 0 dB. More importantly, $a_i = 10$ dB is the limit of the range where the two mass point approximation to $I_{FC}(a_i)$ is accurate. Numerical convergence issues in the gradient search for I_{FC} at higher a's and more mass points thwarted our ambitions to evaluate the bound for the KI noise model up to 0 dB SNR, but these issues should be resolvable with some additional effort. Figure 6 plots the lower bound on the range accuracy Δr for both the KI and SW noise models for -20 dB

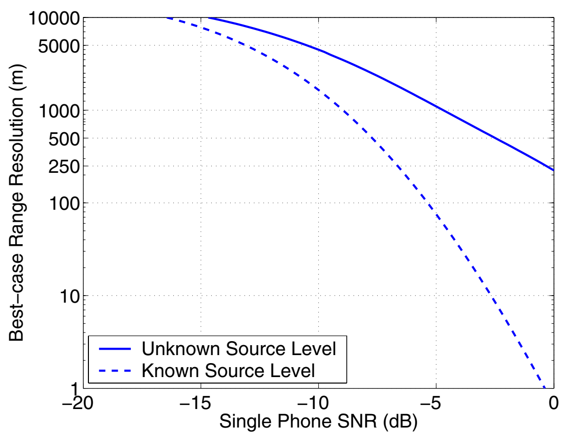

Fig. 5. Best-case range accuracy vs. SNR for both known and unknown source levels in SW noise model.

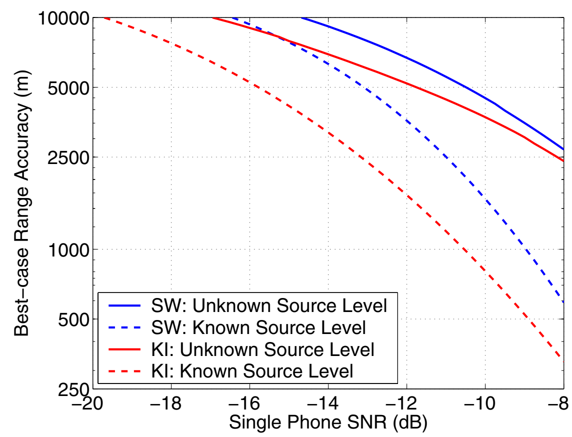

Fig. 6. Best-case range accuracy vs. SNR for both SW and KI noise models with known and unknown source levels.

$\leq$ SNR ≤ -8 dB. The noise covariance $\mathbf{K_{NN}}$ for the KI model was computed using OASES. The KI bound doesn't reach 250 m for these SNRs, but at the relatively large nominal value of 2500 m, there is a 13 dB increase in the SNR needed for ASP if the average source level instead of the exact source level is known. Comparing the SW curves (blue lines) with the KI curves (red lines), the KI noise model has a lesser bound on Δr for this region. The KI noise model has larger a_i for mode 1 than any of the modes in the SW noise model, at the expense of smaller a_i values for many of the other modes. The gain in information from mode 1 outweighs the loss in the other modes, giving the KI model more information than the SW model, and consequently a lower bound on Δr.

IV. Conclusion

Incorporating the memoryless Rayleigh-fading channel model from [8] provides a more realistic and tighter bound on passive sonar performance than prior results presented in [6]. The bounds shown in Figs. 5 and 6 should be taken as indicative rather than absolute due to the small errors in approximating $I_{FC}(a_i)$ with only two mass points. More accurate numerical evaluation of the fading channel capacity will provide better bounds and larger SNRs than presented here.

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